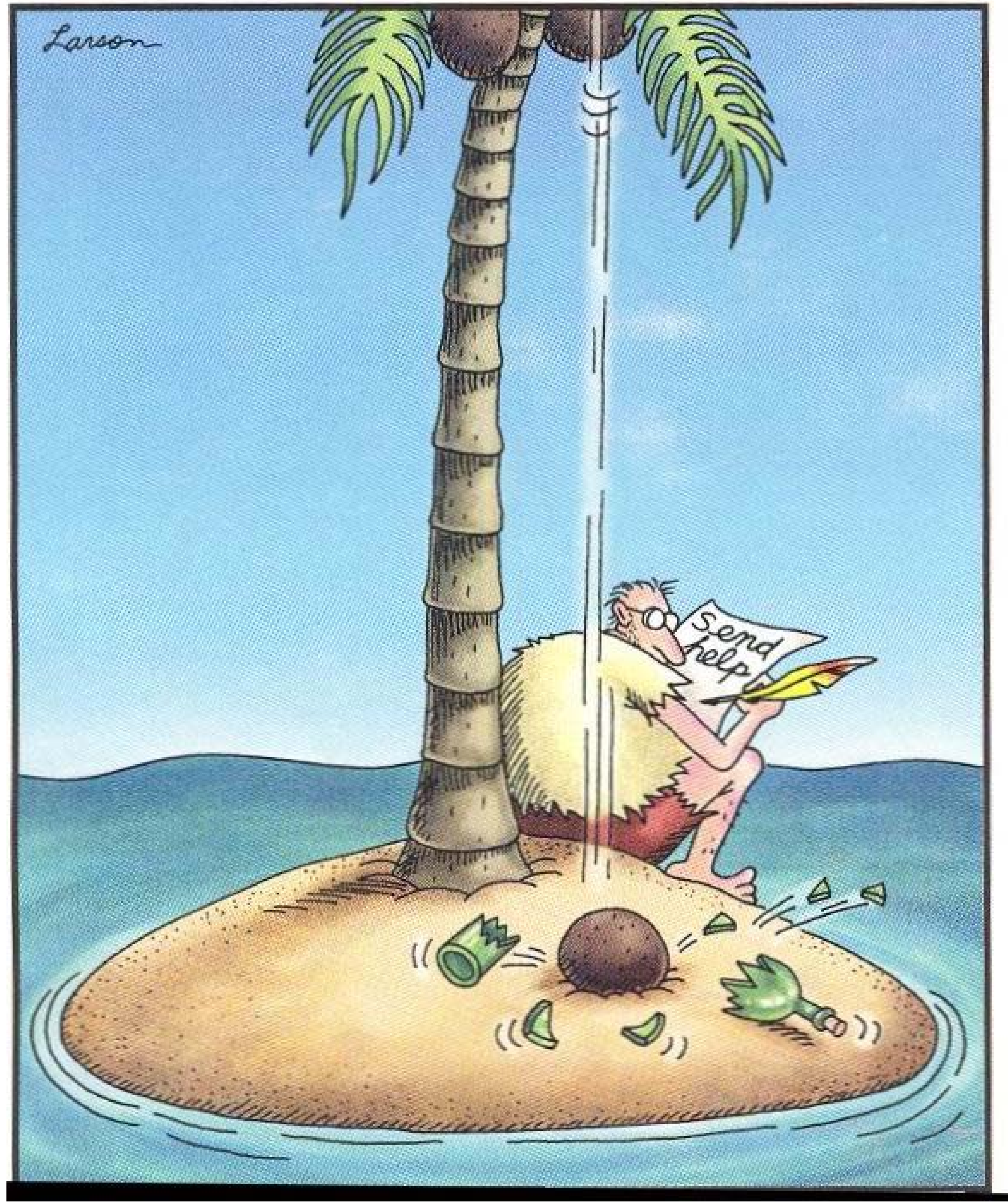
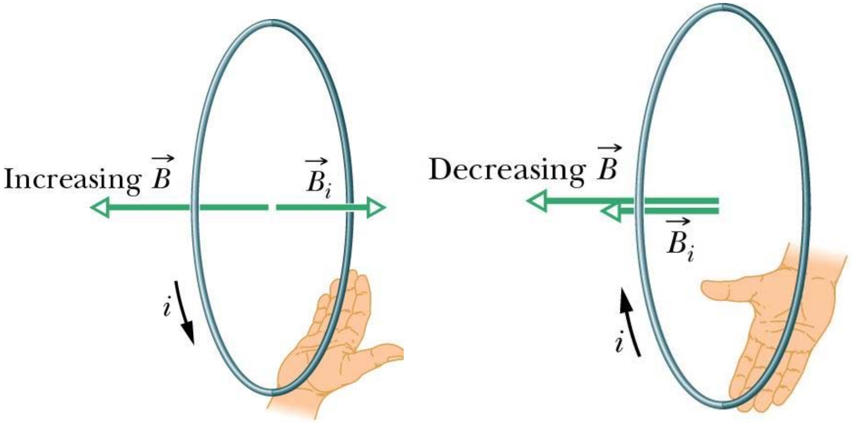
AC Circuits



Review

- Magnetic flux
$$\Phi_B = \int \vec{B} \bullet d\vec{A}$$
- Faraday's law (one loop) for emf (E) (induced voltage)
$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$
- Faraday's law (N loops)
$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$
- Lenz's law – induced emf gives rise to a current whose *B* field opposes the change in flux that produced it



Review for Inductor

- Inductor is a device used to produce and store a desired B field (e.g. solenoid)
- A current i in an inductor with N turns produces a magnetic flux, Φ_B, in its central region
- Inductance, L is defined as

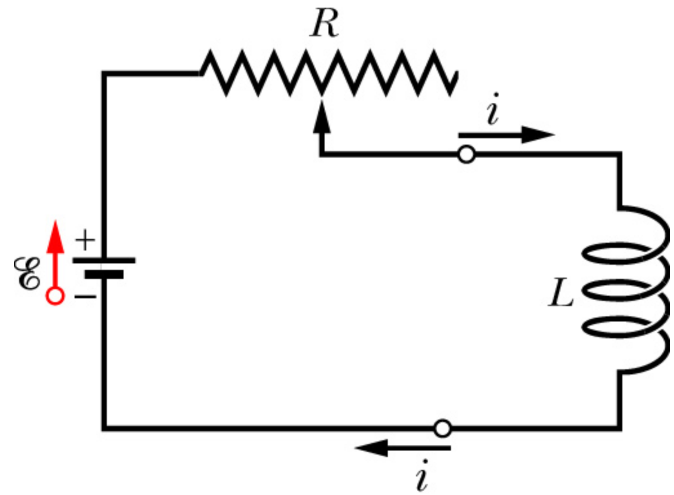
$$L = \frac{N\Phi_B}{i}$$

- Inductance per unit length of a solenoid
  - Depends only on geometry of device (like capacitor)

$$\frac{L}{l} = \mu_0 n^2 A$$

Inductor in a circuit

- A changing current in a coil generates a self-induced emf, ε_L in the coil
- Process is called self-induction
- Change current in coil using a variable resistor, ε_L will appear in coil only while the current is changing



$$L = \frac{N\Phi_B}{i}$$

$$\mathcal{E}_L = -N\frac{d\Phi_B}{dt} = -\frac{d(N\Phi_B)}{dt} = -\frac{d(Li)}{dt} = -L\frac{di}{dt}$$

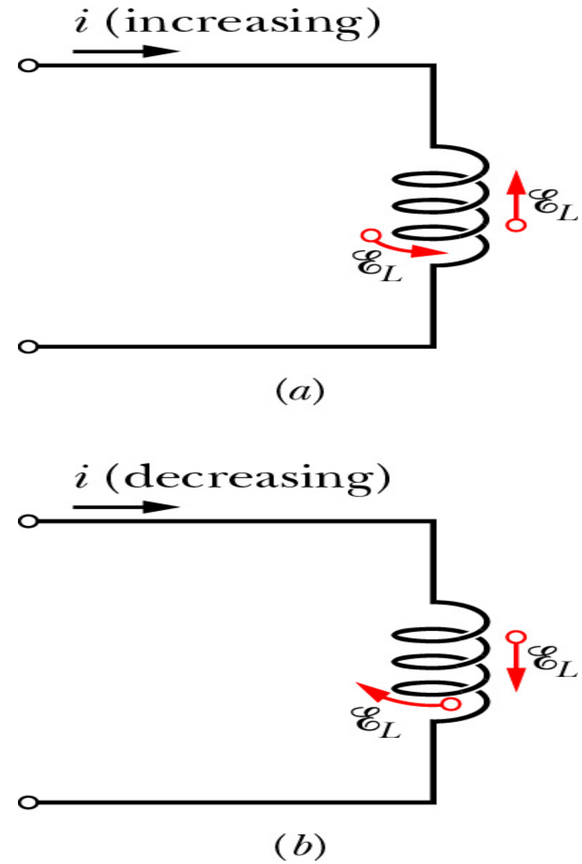
EMF of an Inductor

- Induced emf only depends on rate of change of current, not its magnitude
- Direction of ε_L follows Lenz's law and opposes the change in current
- Self-induced V_L across inductor
  - Ideal inductor
  - Real inductor (like real battery) has some internal resistance

$$\mathcal{E}_L = -L\frac{di}{dt}$$

$$V_L = \mathcal{E}_L$$

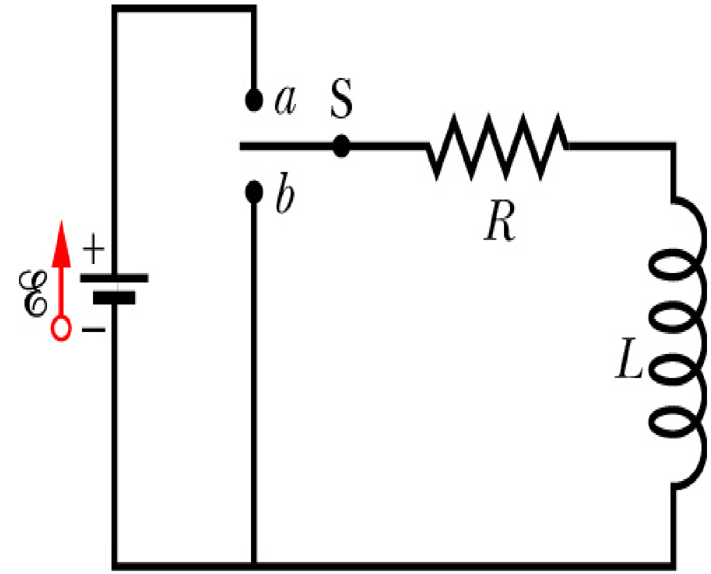
$$V_L = \mathcal{E}_L - iR$$



RL circuit

- RL circuit is a resistor and inductor in series
- Close switch to point a
  - Initially *i* is increasing through inductor so ε_L opposes rise and *i* through *R* will be

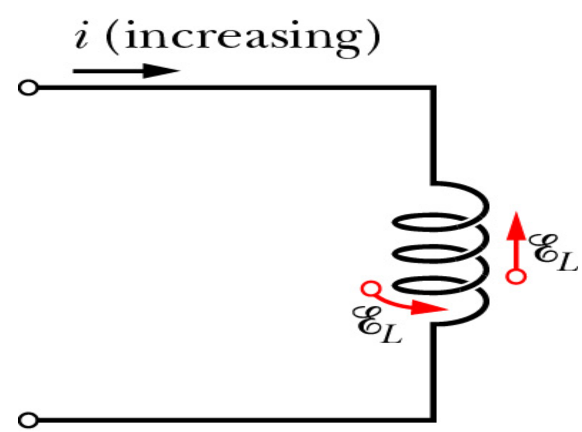
$$i < \mathcal{E}/R$$



$$\mathcal{E}_L = -L\frac{di}{dt}$$

  - Long time later, *i* is constant so ε_L=0 and *i* in circuit is

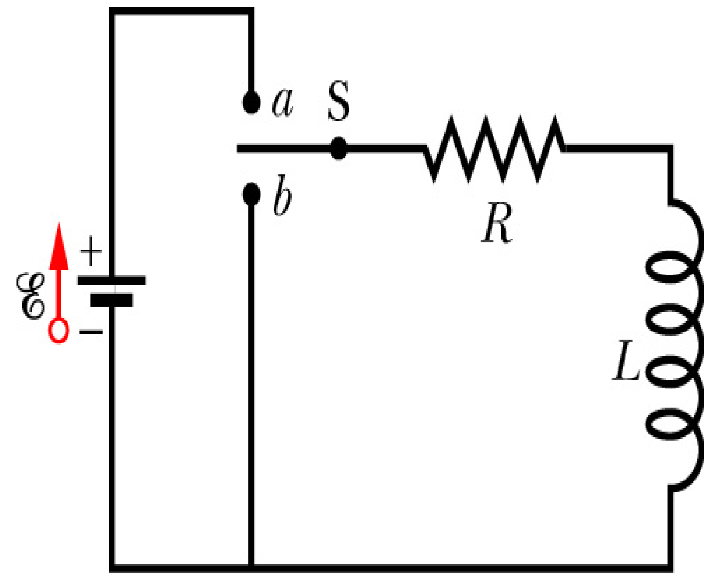
$$i = \mathcal{E}/R$$

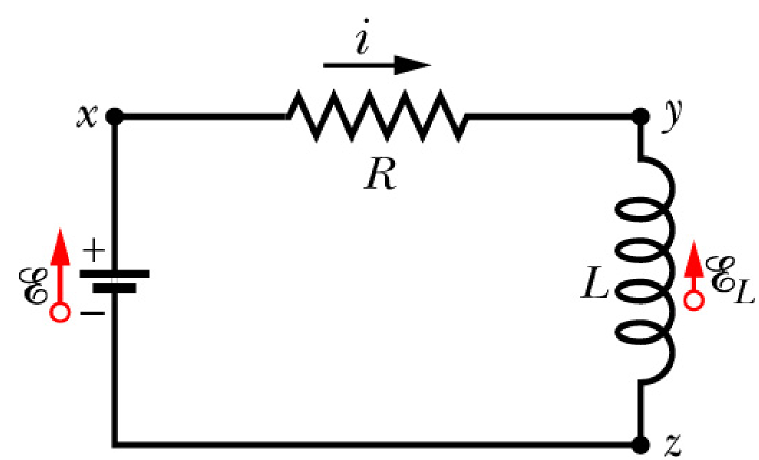


RL circuit

- Initially an inductor acts to oppose changes in current through it
- Long time later inductor acts like ordinary conducting wire
- Apply loop rule right after switch has been closed at a

$$\mathcal{E} - iR - L\frac{di}{dt} = 0$$





Solution RL circuit equation

- Assume $i = Ae^{\alpha t} + B$ and replace in $\mathcal{E} - iR - L\frac{di}{dt} = 0$

$$\Rightarrow \quad \text{☜} - RAe^{\alpha t} - BR - L\alpha Ae^{\alpha t} = 0 \qquad \text{(a)}$$

- But boundary conditions:
  - At t=0, $i = 0$
  - At t=∞, $i = \varepsilon / R$

$$\Rightarrow \quad A = -B \qquad \alpha < 0\,; \qquad B = \text{☜} / R$$

(b)

- Solving (a) and (b):

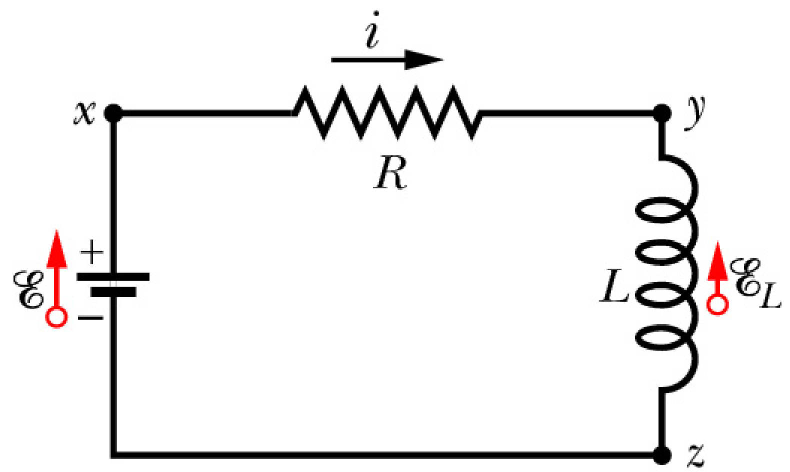
$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right)$$

$$\tau_L = \frac{L}{R}$$

Solving RL circuit equation

- Differential equation similar to capacitors

$$\mathcal{E} = iR + L\frac{di}{dt}$$



- Solution is

$$i = \frac{\varepsilon}{R}\left(1 - e^{-t/\tau_L}\right)$$

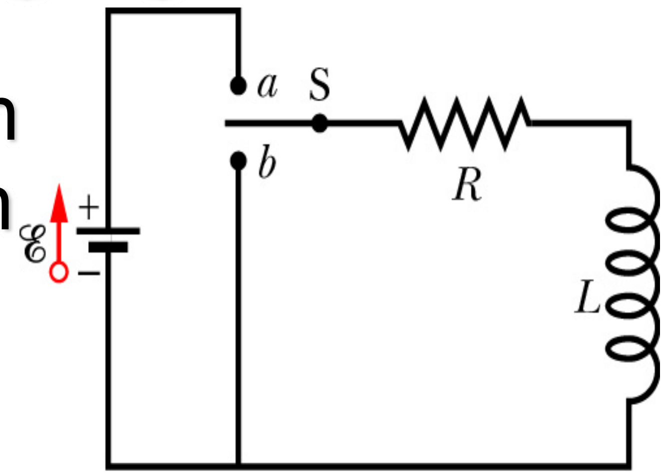
- Inductive time constant is

$$\tau_L = \frac{L}{R}$$

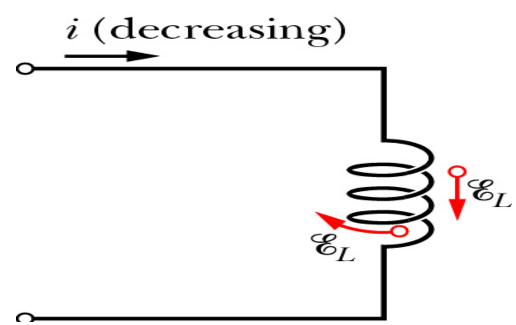
- Satisfies conditions:
  - At t=0, $i = 0$
  - At t=∞, $i = \varepsilon/R$

Charging and discharging RL circuit

- Now move switch to position b so battery is out of system
- Current will decrease with time and loop rule gives



$$iR + L\frac{di}{dt} = 0$$



- Solution is

$$i = \frac{\varepsilon}{R}e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$$

- Satisfies conditions
  - At t=0, $i = i_0 = \varepsilon/R$
  - At t=∞, $i = 0$

RL circuits Summary

- Circuit is closed (switch to “a”)

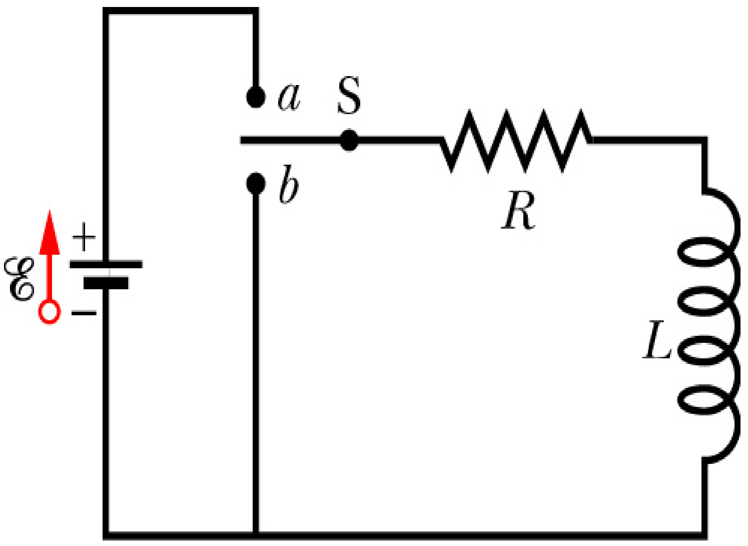
$$i = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau_L}\right)$$

- Circuit is opened (switch to “b”)

$$i = \frac{\mathcal{E}}{R} e^{-t/\tau_L} = i_0 e^{-t/\tau_L}$$

- Time constant is

$$\tau_L = \frac{L}{R}$$

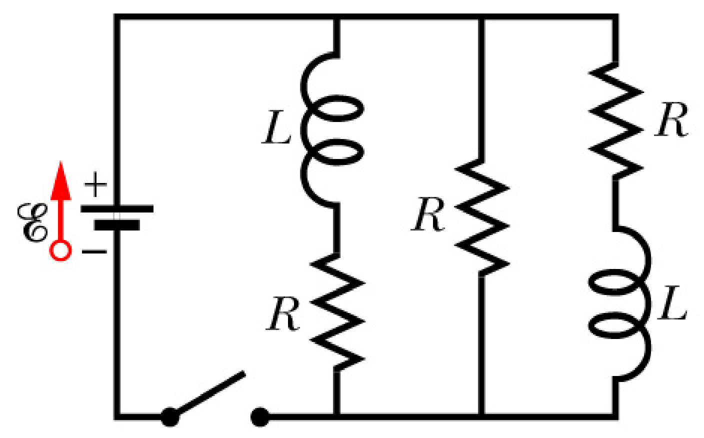


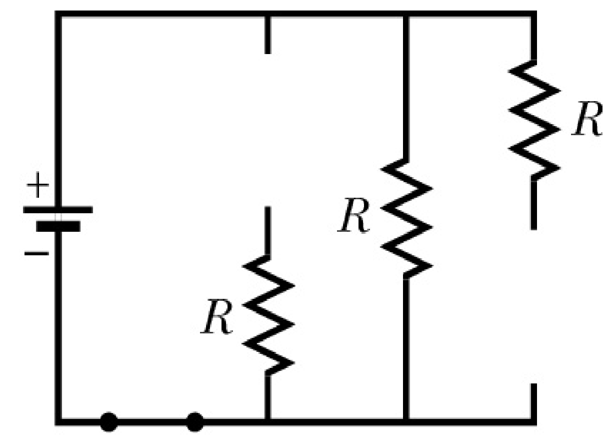
Switch at a, current through inductor is:

- Initially $i = 0$ (acts like broken wire)
- Long time later $i = \varepsilon/R$ (acts like simple wire)

Exercise

- Have a circuit with resistors and inductors
- What is the current through the battery just after closing the switch?
- Inductor oppose change in current through it
- Right after switch is closed, current through inductor is 0
- Inductor acts like broken wire





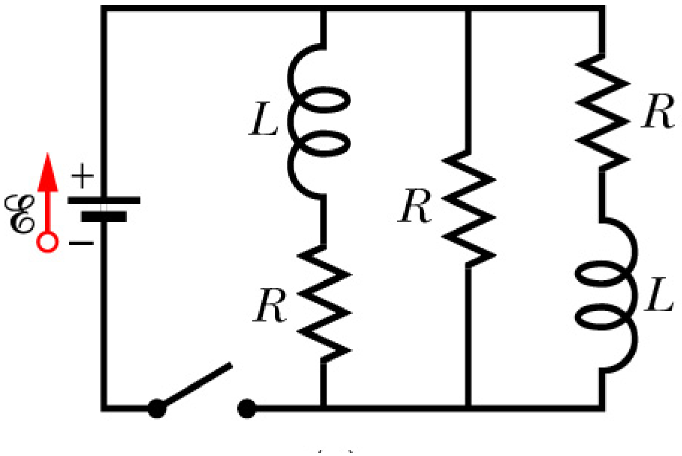
Exercise

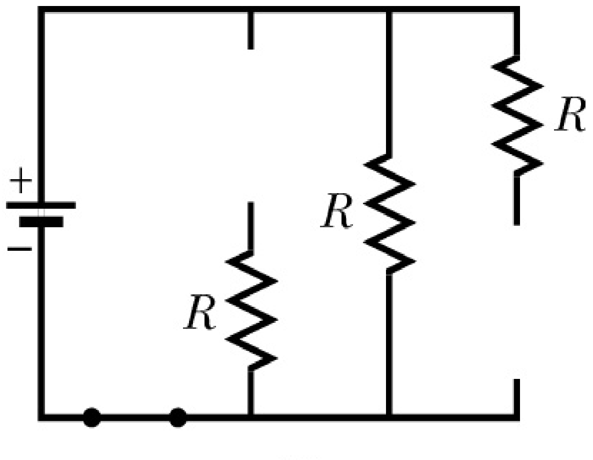
- Apply loop rule

$$\mathcal{E} - iR = 0$$

- Immediately after switch closed, current through the battery is

$$i = \frac{\mathcal{E}}{R}$$



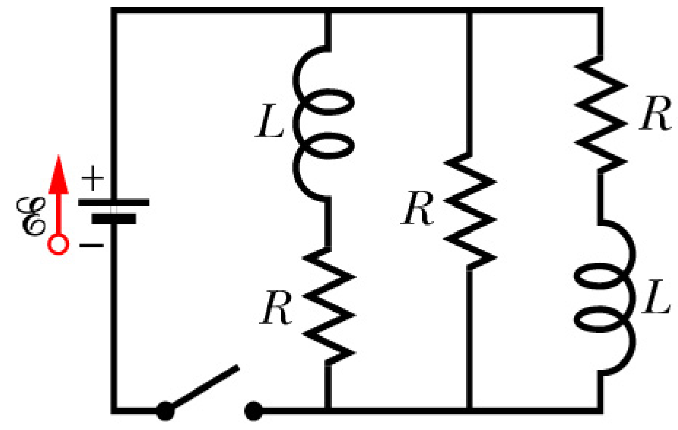


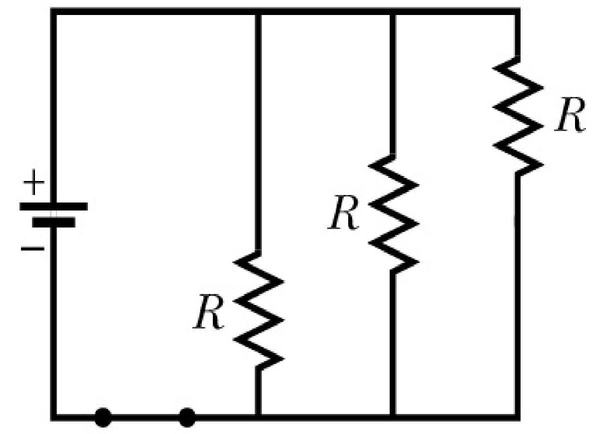
Exercise

- What is the current through the battery a long time after the switch has been closed?
- Currents in circuit have reached equilibrium so inductor acts like simple wire
- Circuit is 3 resistors in parallel

$$i = \frac{E}{R_{eq}}$$

$$R_{eq} = \frac{R}{3}$$

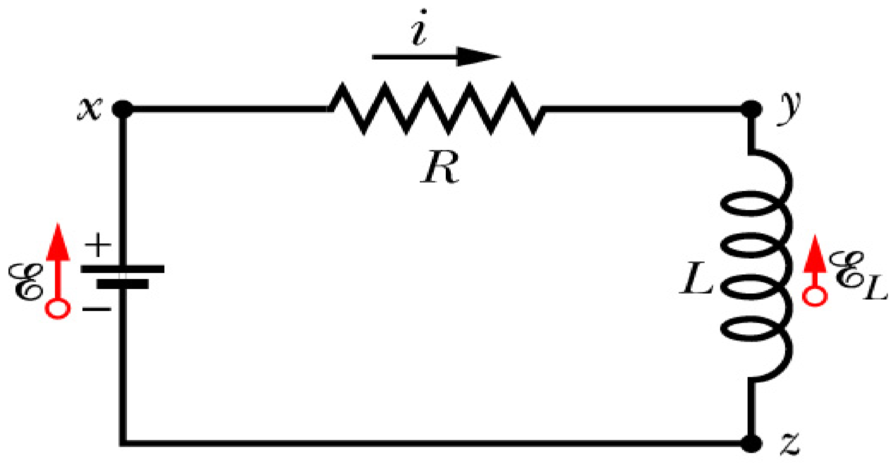




(c)

Energy stored in an inductor

- How much energy is stored in a *B* field?
- Conservation of energy expressed in loop rule



$$\mathcal{E} = L\frac{di}{dt} + iR$$

- Multiply each side by *i*

$$\mathcal{E}i = Li\frac{di}{dt} + i^2R$$

- $P = i\varepsilon$ is the rate at which the battery delivers energy to rest of circuit
- $P = i^2R$ is the rate at which energy appears as thermal energy in resistor

Energy stored in an inductor

- Middle term is rate at which energy dU_B/dt is stored in the B field

$$\frac{dU_B}{dt} = Li\frac{di}{dt}$$

- Energy stored in magnetic field

$$U_B = \frac{1}{2}Li^2$$

- Similar to energy stored in electric field

$$U_E = \frac{1}{2}\frac{q^2}{C}$$