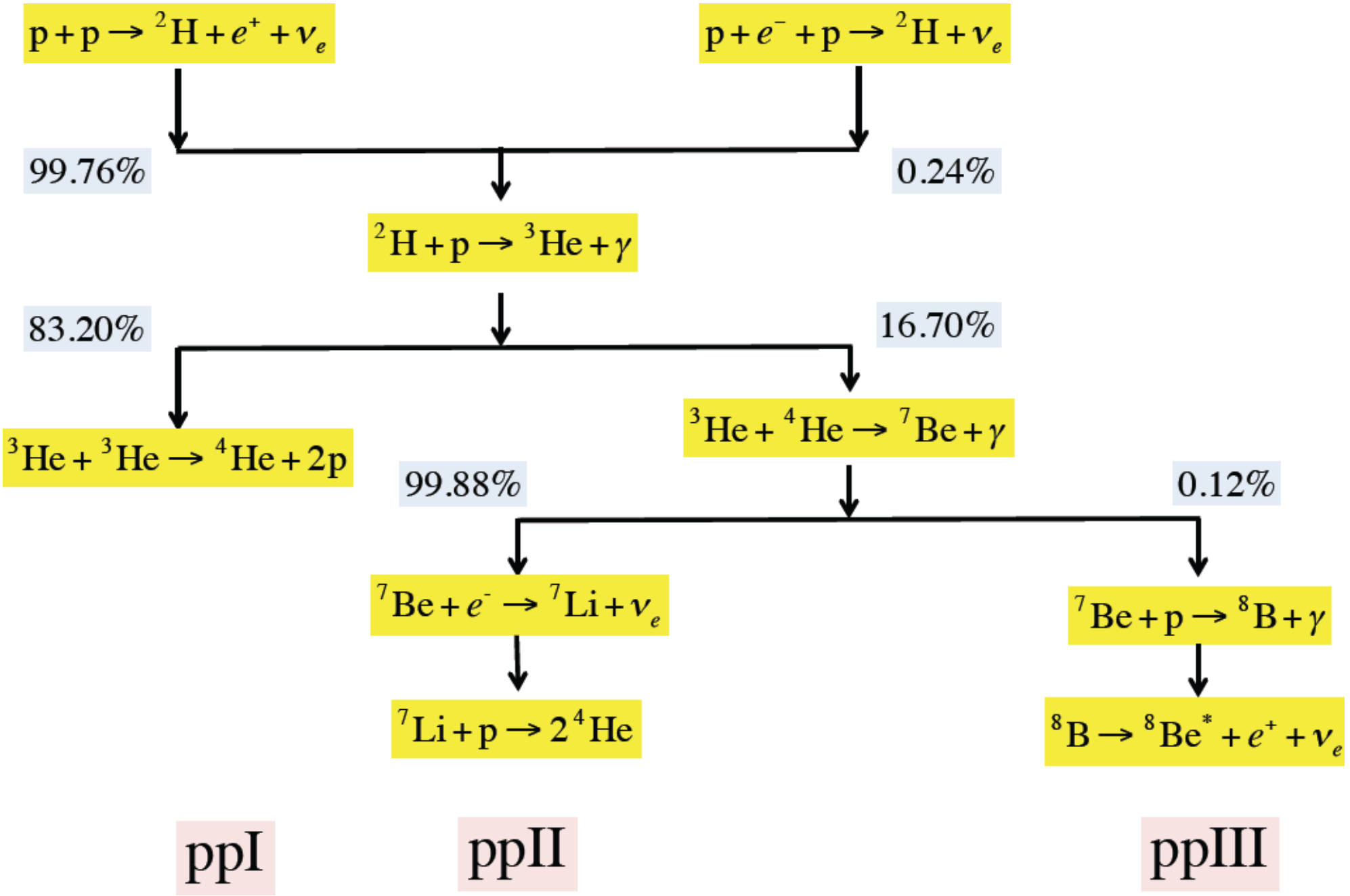
10 – pp-chain and CNO cycle

pp chain reaction in the Sun



10.1 - Hydrogen burning – ppI chain

As a star forms from a giant cloud in the interstellar medium (protostar), density and temperature increase in its center. Fusion of hydrogen (^{1}H) is the first long term nuclear energy source that can ignite. With only hydrogen available (for example, a first generation star) the ppI chain (see previous slide) is the only possible sequence of reactions. (other reaction chains require catalyst nuclei). The ppI chain proceeds as follows:

Step 1: $p + p \rightarrow {}^2He$ This does not work because ^{2}He is unstable

$$p + p \rightarrow d + e^+ + \nu_e \qquad (10.1)$$

Step 2: $$d + p \rightarrow {}^3He \qquad (10.2)$$

$d + d \rightarrow {}^4He$ This does not work because the d abundance is too low. d + p leads to rapid destruction of d.

Step 3: ${}^3He + p \rightarrow {}^4Li$ Does not work because ^{4}Li is unstable

${}^3He + d \rightarrow {}^4He + n$ Does not work because d abundance is too low

$$^3He + {}^3He \rightarrow {}^4He + 2p \qquad (10.3)$$

Last reaction is OK because Y_{3He} gets large as there is no other rapid destruction.

10.1.1 - S-factors for the pp chain

All reactions in the pp-chain are charge-dependent. The cross section has the form given by Eq. (9.49) which can be cast in the form where

$$\sigma \propto \frac{1}{E} e^{-\frac{b}{\sqrt{E}}} \qquad (10.4)$$

$$b = 31.28\, Z_1 Z_2 A^{1/2} \sqrt{\text{keV}} \qquad (10.5)$$

$$A = \frac{A_1 A_2}{A_1 + A_2} \qquad (10.6)$$

Typical units for S(E) are keV barn.

As we have written in Eq. (9.55), the reaction rate samples very low energy cross sections, or very low S-factors, through the Gamow Window E_0 energy being very small. Then the integrand below around the Gamow window can be approximated by a Gaussian with **energy width ΔE**:

$$\langle \sigma v \rangle = \sqrt{\frac{8}{\pi\mu}} (kT)^{-3/2} \int \sigma(E) E\, e^{-\frac{E}{kT}}\, dE = \sqrt{\frac{8}{\pi\mu}} (kT)^{-3/2} \int S(E) e^{-\left(\frac{b}{\sqrt{E}} + \frac{E}{kT}\right)} dE$$

(10.7)

where

$$e^{-\left(\frac{b}{\sqrt{E}} + \frac{E}{kT}\right)} \sim e^{-\frac{1}{2}\left(\frac{E - E_0}{\Delta E/2}\right)^2} \qquad (10.8)$$

$$\Delta E = \frac{4}{\sqrt{3}} \sqrt{E_0 kT} = 0.23682 \left(Z_1^2 Z_2^2 A\right)^{1/6} T_9^{5/6}\ \text{MeV} \qquad (10.9)$$

which leads to (9.55) after integration, as proven in the next two slides.

S-factors at very low energy

The standard method for estimating such an integral is to find the energy that maximizes the exponential and expand around this peak in the integrand. This corresponds to solving

$$\frac{d}{dE}\left(\frac{b}{\sqrt{E}}+\frac{E}{kT}\right)=0 \qquad \Rightarrow \qquad b=\frac{2E_0^{3/2}}{kT} \tag{10.10}$$

Now carrying out a Taylor expansion around E_0:

$$f(E)=\left(\frac{b}{\sqrt{E}}+\frac{E}{kT}\right)\sim f\left(E_0\right)+\left(E-E_0\right)\left(\frac{df}{dE}\right)_{E_0}+\frac{1}{2}\left(E-E_0\right)^2\left(\frac{d^2f}{dE^2}\right)_{E_0}+\cdots \tag{10.11}$$

$$\left(\frac{df}{dE}\right)_{E_0}=0 \qquad \Rightarrow \qquad f(E)\sim f\left(E_0\right)+\frac{1}{2}\left(E-E_0\right)^2 f''(E_0) \tag{10.12}$$

Thus

$$\begin{aligned}\langle\sigma v\rangle &= \sqrt{\frac{8}{\pi\mu}}(kT)^{-3/2}\int S(E)e^{-f\left(E_0\right)}e^{-\frac{1}{2}\left(E-E_0\right)^2 f''(E_0)}dE \\ &\sim \sqrt{\frac{8}{\pi\mu}}(kT)^{-3/2}S(E_0)e^{-f\left(E_0\right)}\int_0^\infty e^{-\frac{1}{2}\left(E-E_0\right)^2 f''(E_0)}dE\end{aligned} \tag{10.13}$$

S-factors at very low energy

$$\langle \sigma v \rangle = \frac{4}{\sqrt{\mu}} (kT)^{-3/2} S(E_0) \frac{e^{-f(E_0)}}{\sqrt{f''(E_0)}} \qquad (10.14)$$

But $$f(E_0) = \frac{E_0}{kT} + \frac{b}{\sqrt{E_0}} = \frac{3E_0}{kT} \qquad (10.15)$$

and $$f''(E_0) = \frac{3b}{4E_0^{5/2}} = \frac{3}{2E_0 kT} \qquad (10.16)$$

Thus, we get Eq. (9.55) as promised:

$$\langle \sigma v \rangle = \frac{16}{9\sqrt{3}} \frac{1}{\mu} \frac{1}{2\pi\alpha Z_1 Z_2} S(E_0) \left(\frac{3E_0}{kT} \right)^2$$

S-factors at very low energy

Better (and this is often done for experimental data) one expands S(E) around E = 0 as powers of E to second order:

$$S(E) = S(0) + ES'(0) + \frac{1}{2}E^2 S''(0) \qquad (10.17)$$

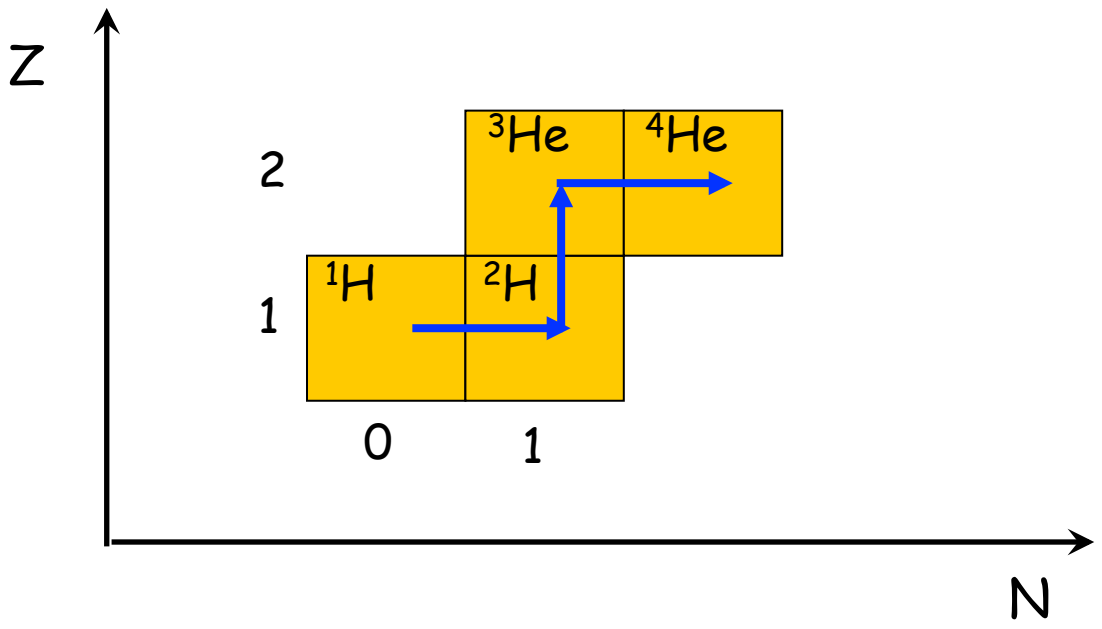
If one integrates this over the Gamov window, one finds that one can use equation of previous slide when **replacing $S(E_0)$ with the effective S-factor S_{eff}**

$$S_{eff} = S(0)\left[1 + \frac{5}{12\tau} + \frac{S'(0)}{S(0)}\left(E_0 + \frac{35}{36}kT\right) + \frac{1}{2}\frac{S''(0)}{S(0)}\left(E_0^2 + \frac{89}{36}E_0 kT\right)\right]$$

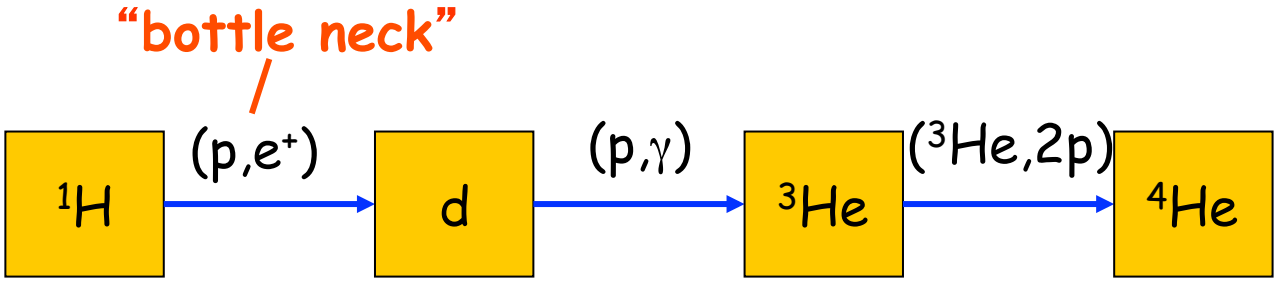
with $\tau = \frac{3E_0}{kT}$ and E_0 as location of the Gamov window. (10.18)

The ppI chain

Let us start by looking at the nuclear chart



In our notation, we call this as a "chain" of reactions, or **reaction chain**:



The first reaction in the chain is **p + p → deuteron (d) + e^+ + ν**. The cross section for it is very small. The other reactions have to wait until this reaction occurs. That is why this reaction is called a "**bottleneck**".

The p + p → d reaction

The **p + p → deuteron (d) + e^+ + ν** cross section is very small because the protons have to overcome their mutual repulsion, tunnel to a distance of about 1 fm. Then one of them has to decay to neutron + neutrino and the neutron has to be captured by the proton to form the deuteron. The cross-section for this reaction for protons of energy below 1 MeV is very small, of the order of 10^{-23} b. In fact, the effective energy for this reaction in the Sun is about 20 keV and this value is much lower.

Another possible reaction forming the deuteron is p + e^- + p → d + n, but the frequency with which it occurs is about 400 times smaller than the pp reaction.

The S-factor for the pp reaction is **$S_{pp}(0) = 4 \times 10^{-25}$ MeV b** and the reaction rate for it is found to be

$$r_{pp} = \frac{1}{2} n_p^2 \langle \sigma v \rangle_{pp} = \frac{1.15 \times 10^9}{T_9^{2/3}} X^2 \rho^2 \exp\left(-\frac{3.38}{T_9^{1/3}}\right) \quad \text{cm}^{-3}\text{s}^{-1} \quad (10.19)$$

where X is the hydrogen mass fraction. The higher order correction leads to **$S'_{pp}(0) = (dS_{pp}/dE)_{E=0} = 11.2 \times S_{pp}(0)$ MeV^{-1}**. See Reviews of Modern Physics 85, 195 (2011).

The p + p → d reaction

The rate of change for proton number because of the pp reaction is given by the

$$\frac{dn_p}{dt} = -\frac{n_p}{\tau_p} \qquad (10.20)$$

where τ_p is the proton half-life due to the reaction. The lifetime is then

$$\tau_p = -\frac{n_p}{dn_p / dt} = \frac{n_p}{2r_{pp}} \qquad (10.21)$$

where the factor two is because two protons are destroyed in the process.

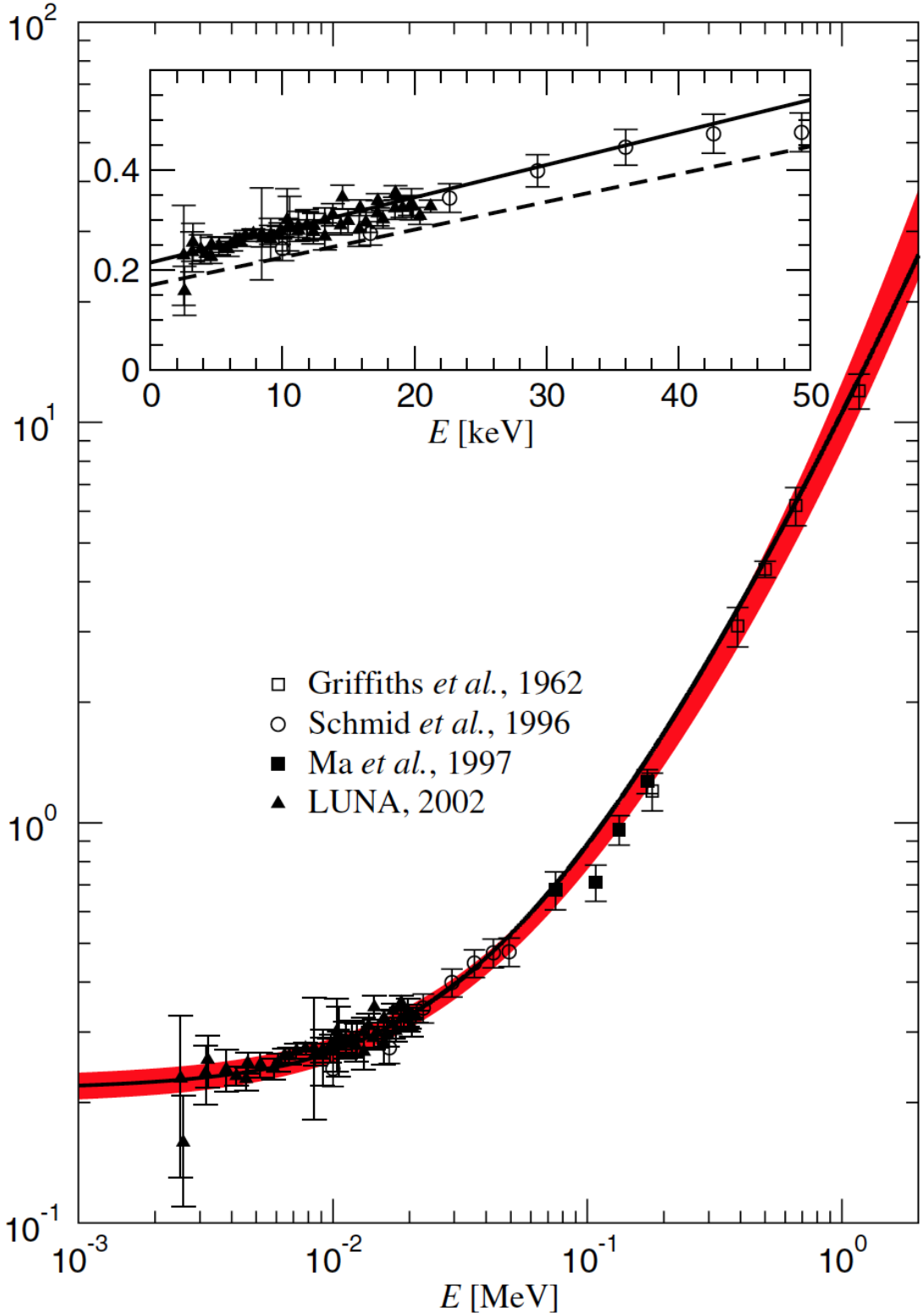
Using $T_9 = 0.15$, $X = 0.5$ and $\rho = 100\ g\ cm^{-3}$ in Eq. (10.19) and using Eq. (10.21) one gets $\tau_p \sim 5 \times 10^9$ years. This is a reasonable estimate for the lifetime of the sun.

Note: it is customary to call S_{pp} by S_{11}, meaning $A = 1$ for each of the two protons. For example, the S-factor for the pd reaction is called S_{12}.

$p + d \rightarrow {}^3He + \gamma$

The S-factor for this reaction is $S_{12}(0) = 0.241$ eV b.

In the figure we show the world data (as of 2011) for the S-factor for this reaction. The curves are theoretical calculations and/or experimental fits. The band is associated to the experimental error bars.



$^3He + {}^3He \rightarrow {}^4He + 2p$

The S-factor for this reaction needs to include higher derivatives of S for a better precision of the reaction rate calculation, as in Eq. (10.17).

In the figure we see that the laboratory screening causes the S-factor to deviate from the desired "bare" S- factor. It is thus necessary to include screening corrections. The table below summarizes the fits for this reaction

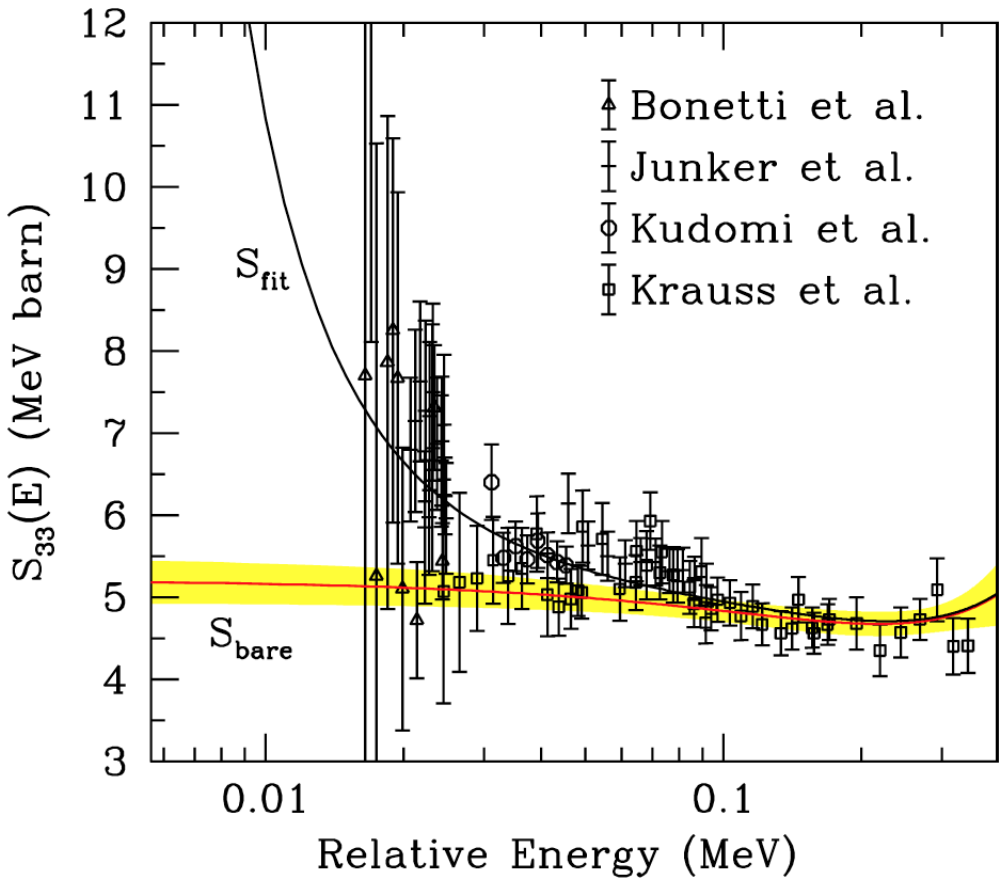


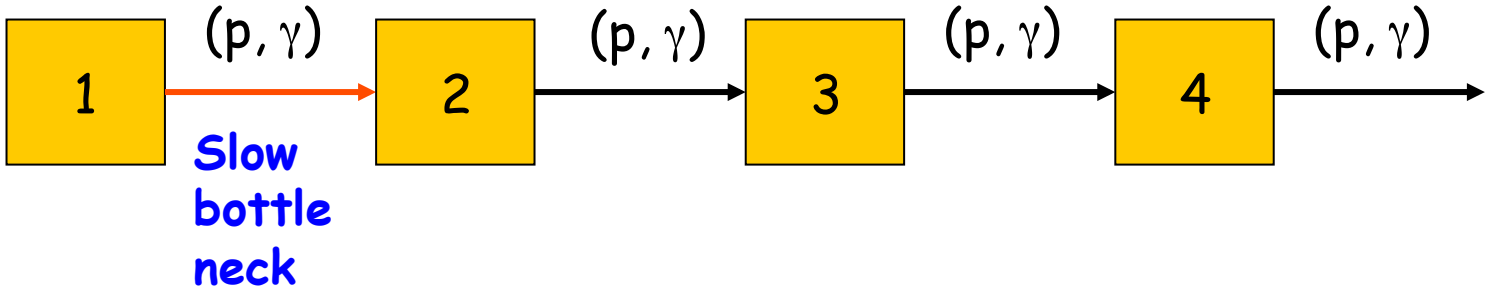
TABLE II. Table of fit parameters and their total errors for constant, linear, and quadratic representations of the bare S factor.

Parameter	Constant	Linear	Quadratic
$S_{33}(0)$ (MeV b)	4.84 ± 0.13	4.95 ± 0.15	5.32 ± 0.23
$S'_{33}(0)$ (b)		-1.06 ± 0.51	-6.44 ± 1.29
$S''_{33}(0)$ (MeV^{-1} b)			30.7 ± 12.2
U_e (eV)	395 ± 50	360 ± 55	280 ± 70
χ^2_{tot}	35.4	34.1	31.8
χ^2_{tot}/dof	0.40	0.39	0.37

Rev. Mod. Phys. 85, 195 (2011).

10.2 - Reaction networks with a bottle-neck

Consider a schematic chain of proton captures shown below:



We will assume that:

- Y_1 ~ const because the depletion is very slow due to the "bottle neck"
- Capture rates are constant (Y_p ~ const because of the large "reservoir", abundant protons in the environment)

The abundance of nucleus 2 evolves according to the equations

$$\frac{dY_2}{dt} = Y_1\lambda_{12} - Y_2\lambda_{23} \qquad (10.22)$$

$$\lambda_{12} = \frac{1}{1+\delta_{p1}} Y_p \rho N_A \langle \sigma v \rangle_{1\to 2} \qquad (10.23)$$

The first term on the rhs of Eq. (10.22) is due to production of 2, the second term due to destruction of 2.

Reaction networks with a bottle-neck

We have assumed that Y_1 ~ const and Y_p ~ const. Hence, after some time Y_2 will then reach an equilibrium value regardless of its initial abundance.

When equilibrium is reached, we get

$$\frac{dY_2}{dt} = Y_1\lambda_{12} - Y_2\lambda_{23} = 0 \quad (10.24) \qquad \text{and} \qquad Y_2\lambda_{23} = Y_1\lambda_{12} \quad (10.25)$$

When an equilibrium condition of this form is reached we call it **steady flow**.

At a later time we get a similar equation for Y_3 :

$$\frac{dY_3}{dt} = Y_2\lambda_{23} - Y_3\lambda_{34} = 0 \quad (10.28)$$

and $Y_3\lambda_{34} = Y_2\lambda_{23}$ (10.29) leading to result for Y_2 $Y_3\lambda_{34} = Y_1\lambda_{12}$ (10.30)

and so on …

In steady flow $Y_i\lambda_{i\,i+1} = \text{const} = Y_1\lambda_{12}$ (10.26) or $Y_i \propto \tau_i$ (10.27)

That is, all abundance Y_i are proportional to the i species destruction rate $\lambda_{i,i+1}$.

Approaching equilibrium

For λ_i ~ const the solution of Eq. (10.22) is

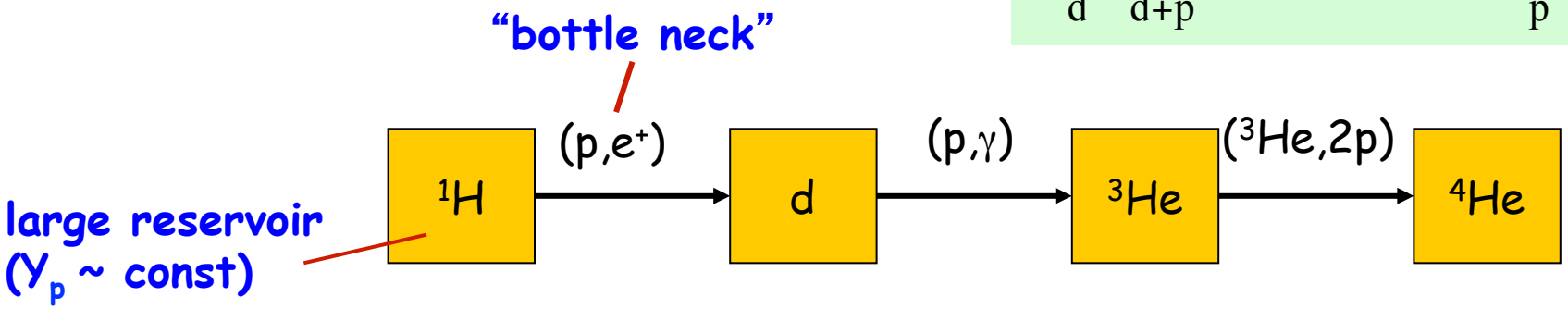
$$Y_2(t) = Y_{2f} - (Y_{2f} - Y_{2i})e^{-t/\tau_2} \quad (10.28)$$

where Y_{2i} is the initial abundance and Y_{2f} the final (equilibrium) abundance.

Regardless of the initial abundance, the equilibrium is reached exponentially within a time equal to the lifetime of the nucleus.

Application to the ppI chain

$$Y_d \lambda_{d+p} = \text{const} = Y_p \lambda_{p+p} \quad (10.29)$$



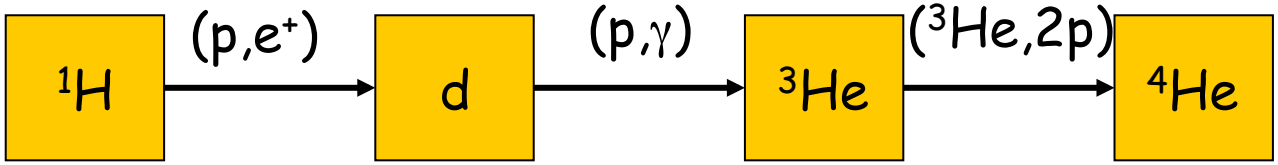
$$\frac{Y_d}{Y_p} = \frac{\lambda_{p+p}}{\lambda_{d+p}} = \frac{Y_p \frac{1}{2}\rho N_A <\sigma v>_{p+p}}{Y_p \rho N_A <\sigma v>_{d+p}}$$

$$\frac{Y_d}{Y_p} = \frac{<\sigma v>_{p+p}}{2<\sigma v>_{d+p}} \quad \begin{matrix} S = 3.8 \times 10^{-19} \text{ eV b} \\ S = 0.24 \text{ eV b} \end{matrix} \quad (10.30)$$

Hence, the equilibrium Y_d is extremely small (~ 4×10^{-18} in the sun). In the sun, the equilibrium is reached within a time set by $N_A <\sigma v>_{pd} = 10^{-2}$ cm^3/s/mole, which yields a deuteron lifetime of

$$\tau_d = 1/(Y_p \rho N_A <\sigma v>_{p+d}) = 2\text{s} \quad (10.31)$$

^{3}He equilibrium abundance



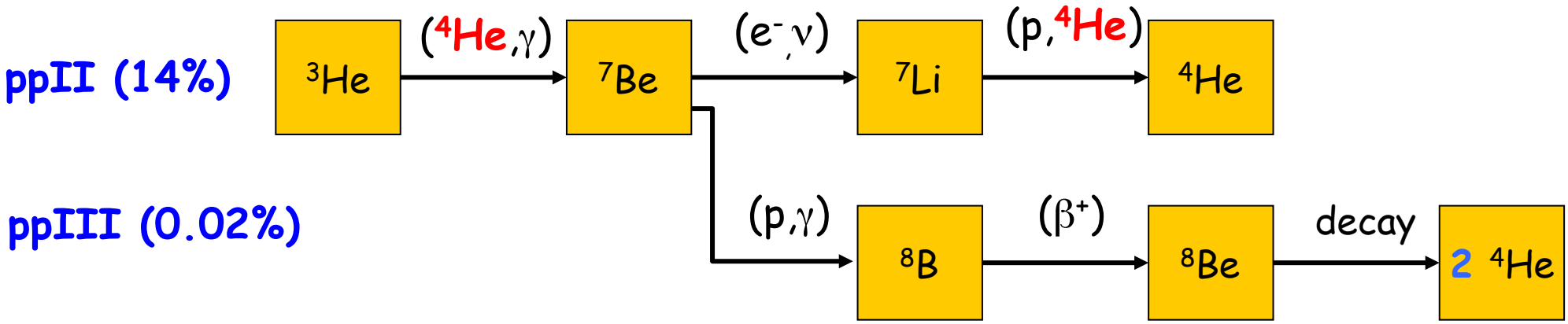
In this case two identical particles fuse and therefore the destruction rate $\lambda_{3He+3He}$ is not constant.

$$\lambda_{3He+3He} = \frac{1}{2} Y_{3He} \rho N_A < \sigma v >_{3He+3He} \qquad (10.31)$$

This depends strongly on Y_{He} itself. The equations cannot be solved with simple approximations and have to be solved numerically. The result is that the ^{3}He abundance is much higher than that of d. Therefore, the fusion ^{3}He + ^{3}He is very probable and ^{4}He can be formed in this way.

The ppII and and ppIII chains

If there is enough ^{4}He in the medium, it will be a **catalyst** of the ppII and ppIII chains to produce even more ^{4}He.



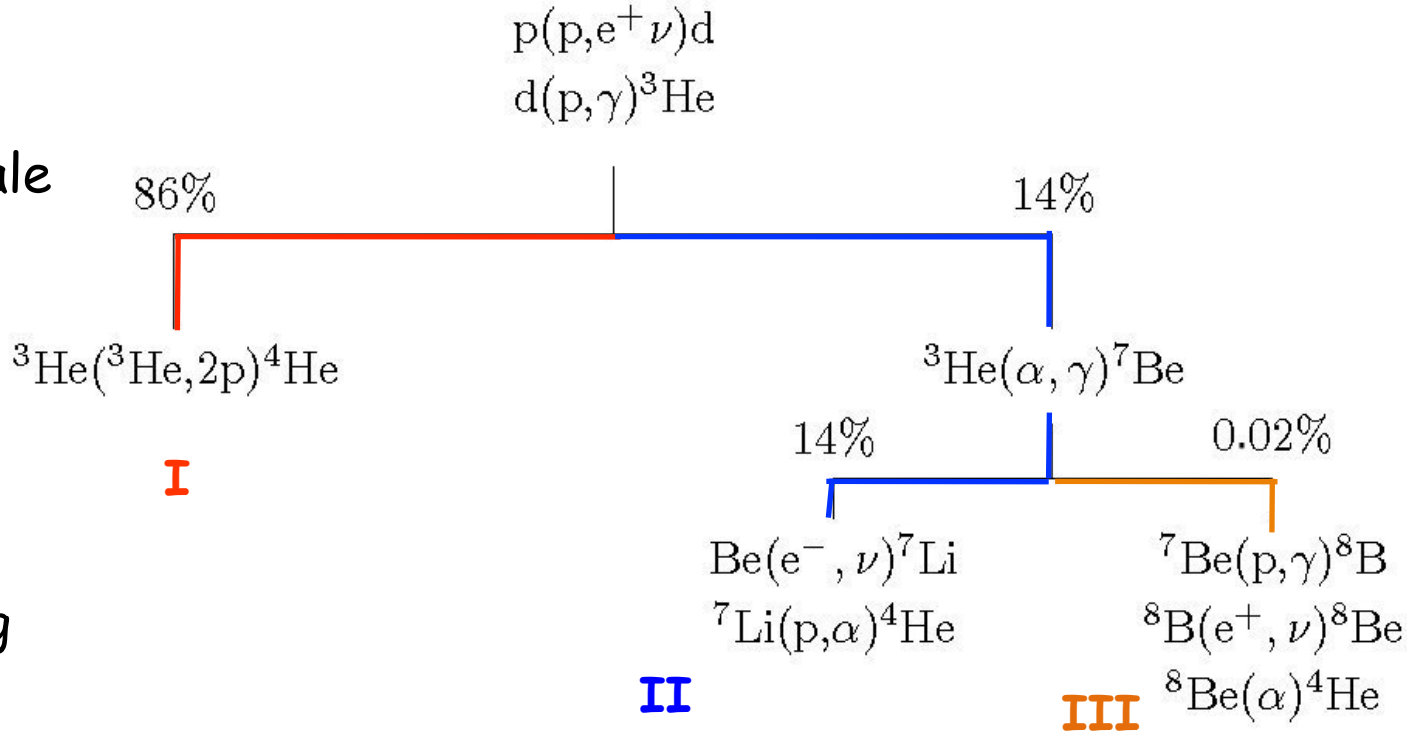
These chains also have to be solved numerically.

Electron capture on ^{7}Be followed by beta decay

A curious and different process occurs in the ppII chain when an electron is captured by ^{7}Be. After the capture of an atomic electron, a neutrino is emitted . The total Q-value for the process is

Q_{EC} = - 862 keV followed by **$Q_{\beta+} = Q_{EC} + 1022 = 160$ keV,** thus becoming energetically possible.

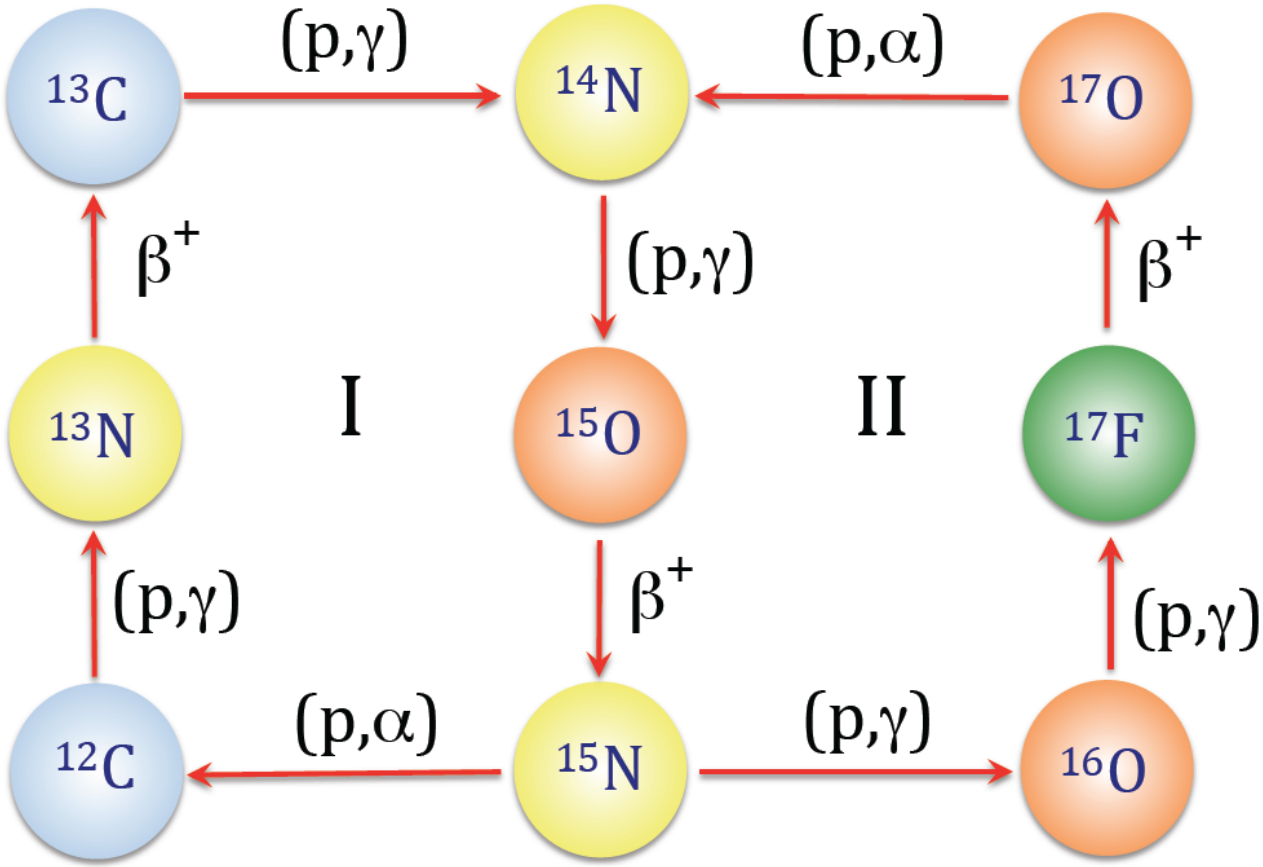
In summary, the timescale of the pp-chain is dominated by the ppI chain. However it produces only ½ of ^{4}He per pp chain. The ppII and ppIII produce the other half thus doubling the production of ^{4}He.



10.3 - The CNO cycle

The CN cycle is part of the CNO cycle. It is the left branch I of the CNO-cycle shown in the figure. The CN cycle was proposed by Hans Bethe in 1938 as a way to use 4 protons to produce ^4He, thus gaining substantial energy per cycle (~ 26 MeV).

The CN cycle is a reaction chain using ^{12}C and ^{14}N as catalysts. Their nuclei remain at the end of the cycle. The ^{12}C nuclei act as seeds that can be reused, despite their very small abundance compared to that of hydrogen.

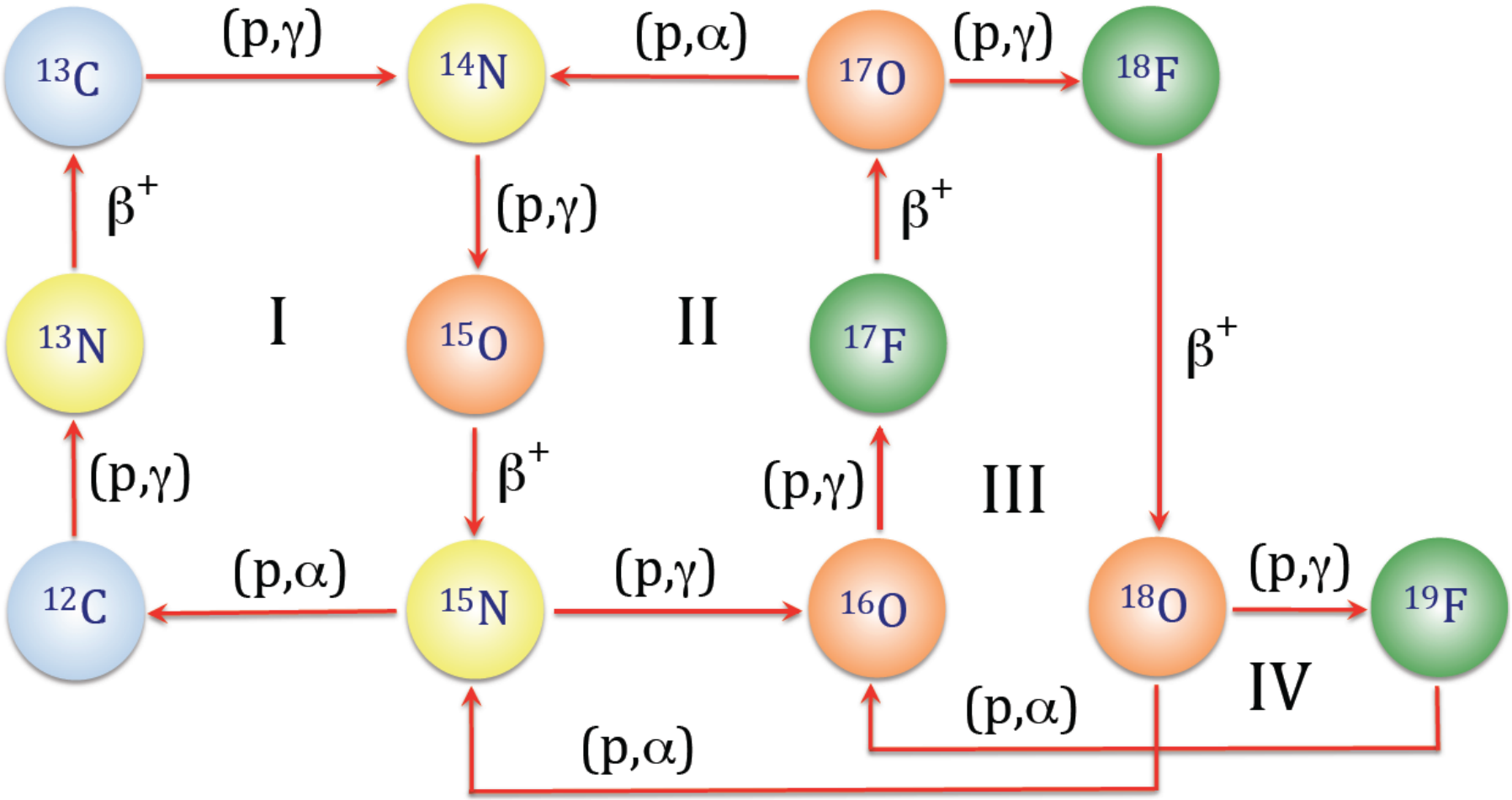


The CN cycle involves the reaction chain

$$^{12}C(p,\gamma)^{13}N(e^{+}\nu_e)^{13}C(p,\ \gamma)^{14}N(p,\gamma)^{15}O(e^{+}\nu)^{15}N(p,\alpha)^{12}C$$

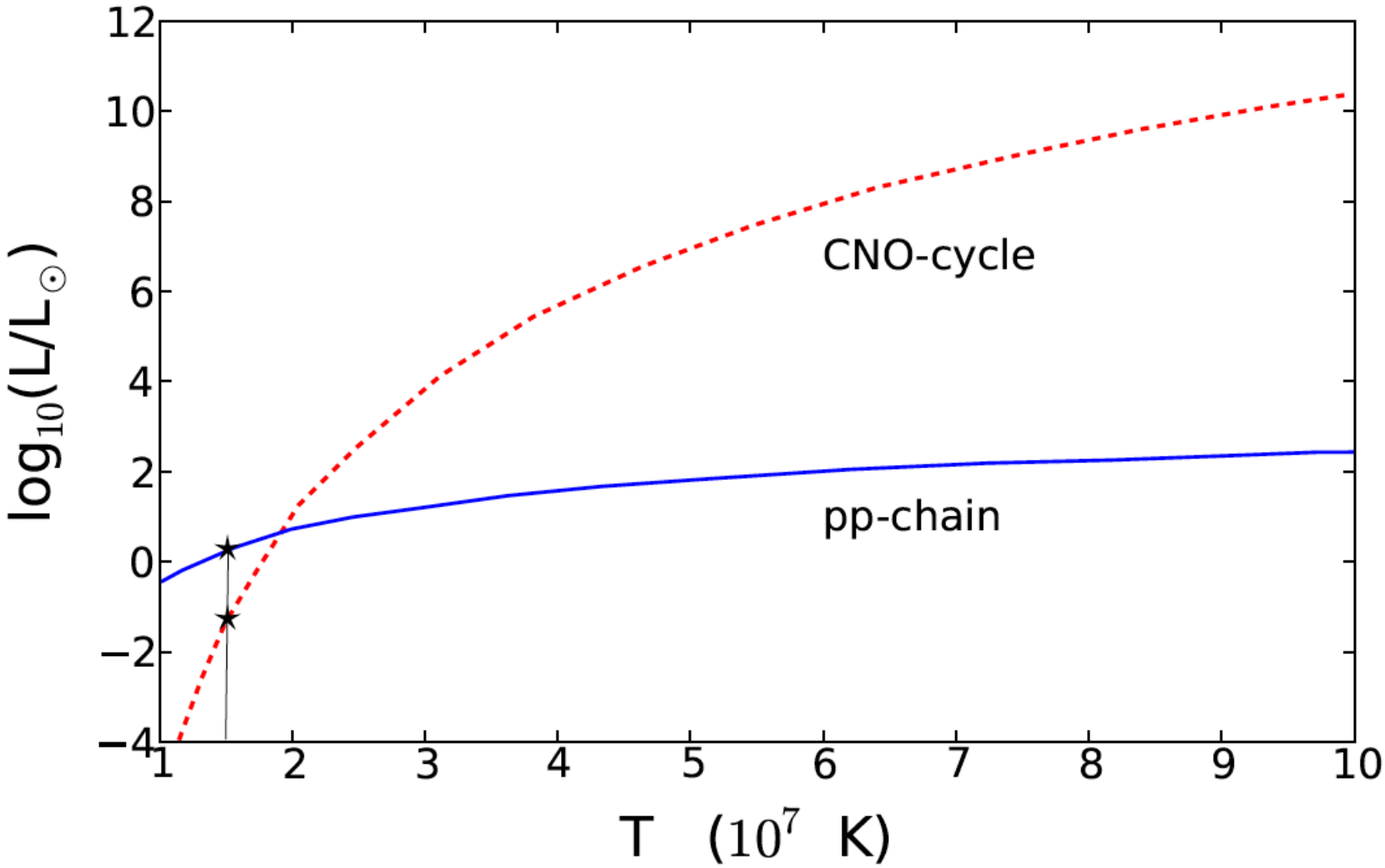
The CNO cycle

The CN cycle can leak from the reaction $^{15}N(p,\gamma)^{16}O$ and form another branch, known as the ON, or CNOII cycle. The nitrogen returns to the initial CN cycle as a ^{14}N nucleus through $^{17}O(p,\alpha)^{14}N$. This is shown in the figure of the previous slide. The ON cycle only works 0.1% of the time compared to the CN cycle because the S-factor for $^{15}N(p,\alpha)^{12}C$ is 1000 times larger than that for $^{15}N(p,\gamma)^{16}O$. A much rarer process happens at higher temperatures, adding a CNOIII cycle through the $^{17}O(p,\gamma)^{18}F$ reaction. All branches of the CNO cycle are shown below.



p-p chain vs. CNO Cycle

The CN cycle operates at higher temperature than the sun. It is relevant for temperatures $T_6 > 20$. But the branches II, III and IV can occur at much larger temperatures ($T \sim 10^8 - 10^9$ K such as (a) hydrogen burning at the accreting surface of a NS, (b) explosive burning on the surface of a WD (novae), (c) in the outer layers, during the shock wave of a supernova explosion. For the sun the pp-chain burning predominates as shown by numerical calculations shown in the figure .



Solar fusion: Summary

A detailed account of solar fusion reactions with cross sections and S-factors for all relevant cases has been reviewed in the article below by an international task force.

REVIEW OF MODERN PHYSICS, VOLUME 83, JANUARY–MARCH 2011

Solar fusion cross sections. II. The pp chain and CNO cycles

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