3- Cosmology - I

3.1 - Matter and Radiation Pressure in the Universe

For an isolated system, $dU + dW = dQ$ (3.1)

where U stands for the **internal energy**, W is the work done by the system and Q is the heat transfer.

Ignoring any heat transfer, $dQ = 0$, and writing $dW = Fdr = pdV$ where F is the force, r is the distance characterizing the size of the system, **p is the pressure** and V is the volume, then

$$dU = -pdV \qquad (3.2)$$

Denoting the **energy density** by **ρ** : $U = \rho V$ (3.3)

$$\Rightarrow \quad \frac{dU}{dt} = \frac{d\rho}{dt}V + \rho\frac{dV}{dt} = -p\frac{dV}{dt} \qquad (3.4)$$

Since **$V \propto r^3$**, then **$(dV/dt)/V = 3(dr/dt)/r$**. Thus,

$$\frac{d\rho}{dt} = -3(\rho + p)\frac{1}{r}\frac{dr}{dt} \qquad (3.5)$$

Matter Dominated Universe

Assuming all energy to be in the form of matter, the relation between matter, *M*, density, ρ, and radius, *r*, means that

$$\rho = \frac{M}{4\pi r^3/3} \qquad (3.6)$$

$$\frac{d\rho}{dt} = \frac{d\rho}{dr}\frac{dr}{dt} = -3\rho\frac{1}{r}\frac{dr}{dt} \qquad (3.7)$$

Comparing Eqs. (3.5) and (3.7) we conclude that

$$\boxed{p = 0} \qquad (3.8)$$

for **matter dominated Universe**.

That is, if no kinetic energy is taken into account, pressure is zero for a system with mass *M*. This is the same pressure as the **ideal gas law for zero temperature**.

Radiation Dominated Universe

Let us consider radiation modes in a cavity based on analogy with a string held fixed at two points separated by a distance L. The possible wavelengths, λ, of a standing wave on the string obey the relation

$$L = \frac{n\lambda}{2} \qquad (3.9)$$

$n = 1, 2, 3, \ldots$. Radiation travels at the velocity of light, so that

$$c = f\lambda = f\frac{2L}{n} \qquad (3.10)$$

where f is the frequency. Planck's formula for the energy of a quantum of radiation with frequency $f = \omega/2\pi$ is $U = \hbar\omega = hf$, where $\hbar = h/2\pi$, and h is **Planck's constant**. Thus,

$$U = \frac{1}{L}\frac{nhc}{2} \sim V^{-1/3} \qquad (3.11)$$

where $V = L^3$ is the volume of a cube of length L. Using Eq. (3.2) the pressure becomes

$$p = -\frac{dU}{dV} = \frac{1}{3}\frac{U}{V} \qquad (3.12)$$

This, together with $\rho = U/V$, yields the **radiation pressure**:

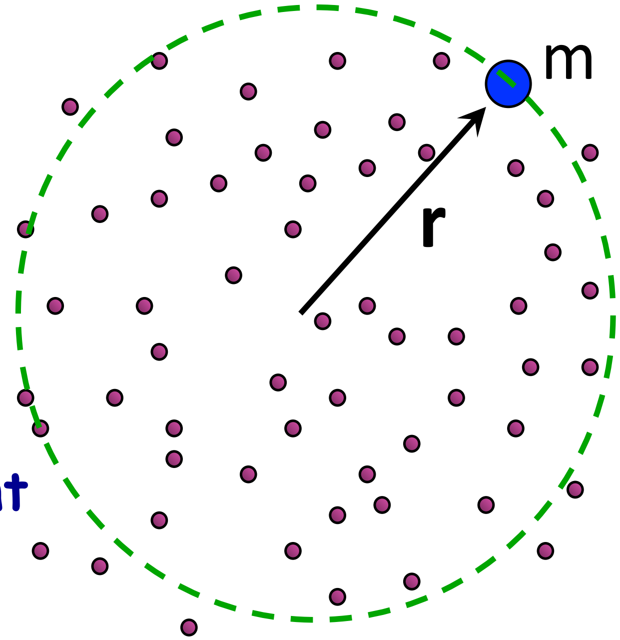
$$\boxed{p = \frac{\gamma}{3}\rho} \qquad (3.13)$$

with $\gamma = 1$ (radiation).
For matter pressure $\gamma = 0$.

3.2 - Friedmann Equation

Birkoff's Theorem: the gravitational force on the particle inside a uniform shell is the same as if the enclosed mass within radius r is localized entirely at the origin, r = 0.

If it is located a distance r from the center of the dust, the total energy E of the particle is then given by



$$E = T + V = \frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r} = \frac{1}{2}mr^2\left(H^2 - \frac{8\pi G}{3}\rho\right) \qquad (3.14)$$

Where $\dot{r}$ = *dr/dt*, *H = (dr/dt)/r* is the **Hubble constant** (see Eq. 2.7), *G* is **Newton's constant**, and we used $M = \rho 4\pi r^3/3$ in the last passage.

Using Eq. (1.3) for the escape velocity, $v_{esc} = [2GM/r]^{1/2} = [(8\pi G/3)\rho r^2]^{1/2}$, Eq. (3.14) can be written as

$$\dot{r}^2 = v_{esc}^2 - k' \qquad (3.15)$$

with $k' = -2E/m$. The constant ***k'* can either be negative, zero or positive,** corresponding to the total energy **E being positive, zero or negative**.

Friedmann Equation

Equation (3.14) is re-arranged as

$$H^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mr^2} \qquad (3.16)$$

Writing the distance in terms of a scale factor ***a*** and a constant length ***s*** as $r(t) = a(t)s$, and defining $k = k'/s^2 = -2E/ms^2$, it follows that

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}} \qquad (3.17)$$

Since $\dot{r}/r = \dot{a}/a$ and $\ddot{r}/r = \ddot{a}/a$. This is the **Friedmann equation**. It specifies the speed of recession of the Universe.

Even though Friedmann equation was derived for matter, it is also true for radiation. Exactly the same equation is obtained from the ***general relativity Einstein field equations.***

The factor *k* can be rescaled so that instead of being negative, zero or positive it takes on the values **−1, 0** or **+1**. In Newtonian mechanics this corresponds to unbound, critical or bound trajectories. From a geometric point of view, this corresponds to an **open, flat or closed Universe.**

3.3 - The Cosmological constant

The acceleration for the Universe is obtained from Newton's second equation, i.e.

$$-G\frac{mM}{r^2} = m\ddot{r} \qquad (3.18)$$

In terms of the density and the scale *a*,

$$\frac{F}{mr} = \frac{\ddot{r}}{r} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\rho \qquad (3.19)$$

Another way to get this equation is to take the time derivative of Eq. (3.17) (valid for matter and radiation)

$$\frac{d}{dt}\dot{a}^2 = 2\dot{a}\ddot{a} = \frac{8\pi G}{3}\frac{d}{dt}\left(\rho a^2\right) \qquad (3.20)$$

Upon using Eq. (3.5) the acceleration equation is obtained as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) = -\frac{4\pi G}{3}\left(1+\gamma\right)\rho \qquad (3.21)$$

which reduces to Eq. (3.19) for the matter equation of state ($\gamma = 0$).

The Cosmological constant

The Universe is unstable to gravitational collapse. Both Newton and Einstein believed that the Universe is static. In order to obtain this Einstein introduced a repulsive gravitational force, called the **cosmological constant**. In order to obtain a possibly zero acceleration, a positive term (conventionally taken as $\Lambda/3$) is added to the acceleration Eq. (3.21) as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3} \qquad (3.22)$$

which, with the proper choice of Λ, can give the required zero acceleration for a static Universe.

The question now is how this repulsive force enters the Friedmann Equation. Identifying the force from

$$\frac{\ddot{r}}{r} = \frac{\ddot{a}}{a} = \frac{F_{\exp}}{mr} = \frac{\Lambda}{3}$$

and using

$$F_{\exp} = \frac{\Lambda}{3}mr = -\frac{dV_{\exp}}{dr}$$

gives the potential energy

$$V_{\exp} = -\frac{1}{2}\frac{\Lambda}{3}mr^2. \qquad (3.23)$$

which is just a **simple repulsive harmonic oscillator**.

The Cosmological constant

Replacing this into the conservation of energy equation,

$$E = T + V = \frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r} - \frac{1}{2}\frac{\Lambda}{3}mr^2 = \frac{1}{2}mr^2\left(H^2 - \frac{8\pi G}{3}\rho - \frac{\Lambda}{3}\right) \qquad (3.24)$$

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}} \qquad (3.25)$$

Equations (3.22) and (3.25) constitute the fundamental equations of motion that are used in all discussions of **Friedmann models of the Universe**.

One often writes the cosmological constant in terms of a vacuum energy density as $\Lambda = 8\pi G\rho_{vac}$ so that the velocity and acceleration equations become

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho + \rho_{vac}\right) - \frac{k}{a^2}} \qquad (3.26)$$

and

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(1+\gamma\right)\rho + \frac{8\pi G}{3}\rho_{vac} \qquad (3.27)$$

3.4 - Matter, Curvature, and Dark Energy

From Eq. (3.25),
$$1 = \frac{8\pi G\rho}{3H^2} - \frac{k}{H^2a^2} + \frac{\Lambda}{3H^2} \quad (3.28)$$

Each of the terms in this equation has special significance. The **mass density** is
$$\Omega_m = \frac{8\pi G\rho}{3H^2} \quad (3.29)$$

The **curvature density** is
$$\Omega_k = -\frac{k}{H^2a^2} \quad (3.30)$$

The **vacuum energy density**, or **dark energy**, is
$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \quad (3.31)$$

Another quantity of interest is the **critical density**, given by
$$\boxed{\rho_{crit} = \frac{3H^2}{8\pi G}} \quad (3.32)$$

in terms of which the mass density can be written as $\Omega_m = \rho/\rho_{crit}$. In terms of the present value of the Hubble parameter the critical density is,

$$\rho_{crit} = 1.88\times10^{-20}\, h^2\,\mathrm{g\,cm^{-3}} \quad (3.33)$$

$$h = \frac{H}{100\ \mathrm{Mpc^{-1}\,s^{-1}}} \approx 0.7 \quad (3.34)$$

Dark Energy

Defining $\Omega = \Omega_m + \Omega_\Lambda$ (3.35)

the Friedmann equation can be rewritten as

$$\boxed{(\Omega - 1)H^2 = \frac{k}{a^2}} \qquad (3.36)$$

so that ***k* = 0, +1, – 1** corresponds to ***Ω* = 1, *Ω* > 1** and ***Ω* < 1.**

The matter density decreases with the radius of the Universe as $\rho(t=0)/\rho(t) = a^3_0/a^3$. Thus, we can write a **mixture of matter and dark energy** by (here, the index "0" means the present value of the variables.)

$$\rho = \rho_m + \rho_\Lambda = \rho_{m0}\left(\frac{a_0}{a}\right)^3 + \rho_\Delta \qquad (3.37)$$

and the Friedmann equation becomes

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 - H^2\Omega_{m0}\left(\frac{a_0}{a}\right)^2 - H^2\Omega_{\Lambda 0} = -\frac{k}{a^2}} \qquad (3.38)$$

Using k =0 (flat Universe), $\Omega_{m0} = 1 - \Omega_{\Lambda 0}$ and, for simplicity $a_0 = 1$ (in appropriate units), we get

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[(1-\Omega_{\Lambda 0})\frac{1}{a^3} + \Omega_{\Lambda 0}\right]$$

Dark Energy

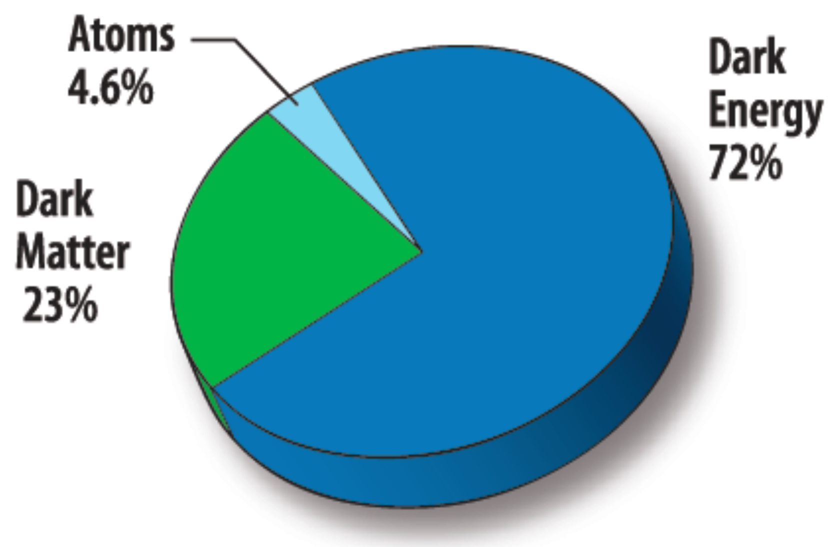
Integrating the last equationover time, with t_0 denoting the age of the Universe, we get

$$H_0 t_0 = \int da \frac{\sqrt{a}}{\sqrt{1-\Omega_{\Lambda 0}+\Omega_{\Lambda 0} a^2}} = \frac{2}{3\sqrt{\Omega_{\Lambda 0}}} \ln\left(\frac{1+\sqrt{\Omega_{\Lambda 0}}}{\sqrt{1-\Omega_{\Lambda 0}}}\right) \qquad (3.39)$$

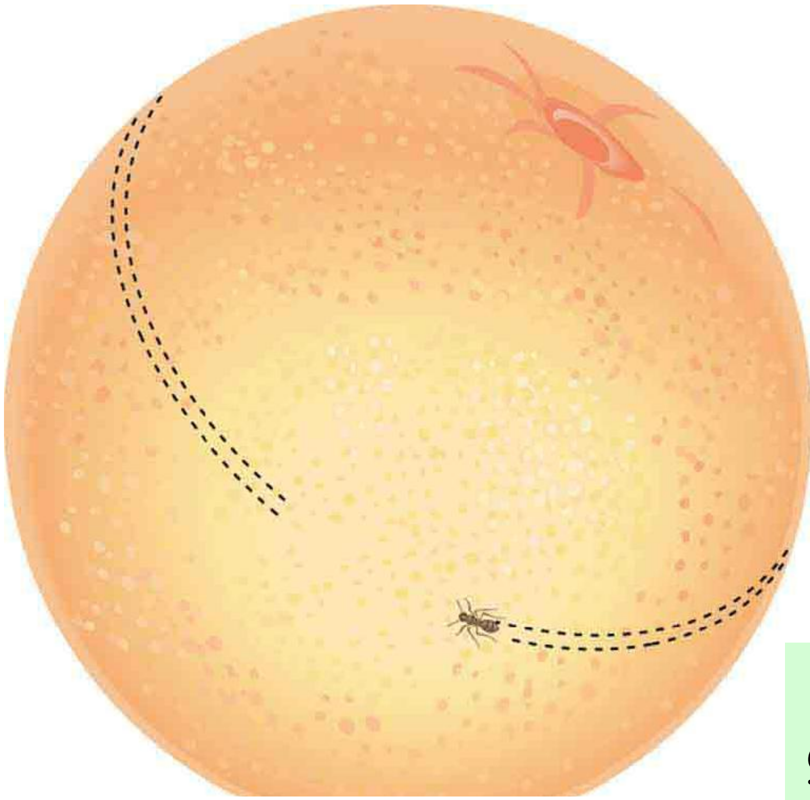
where H_0 is the present value of the Hubble constant. We thus see that, as $\Omega_{\Lambda 0} \rightarrow 1$, then $t_0 \rightarrow \infty$. It is thus necessary to have some matter to keep the age of the Universe finite.

We can turn this argument around. Assuming the age of the Universe to be $t_0 = 13.7$ Gy we get $\Omega_{\Lambda 0} = 0.72$, or $\Omega_{m0} = 0.28$, i.e. only **28% of the Universe is matter** and **72% is dark energy**. Observations also indicates that only 4% of the Universe is baryonic (normal) matter, and that the remaining 24% is in some other still unknown form, a **dark matter**.

Dark matter and dark energy thus compose about 95% of the Universe.



3.5 - Geometry of the Universe



2-dimensional analogy: Surface of a sphere.

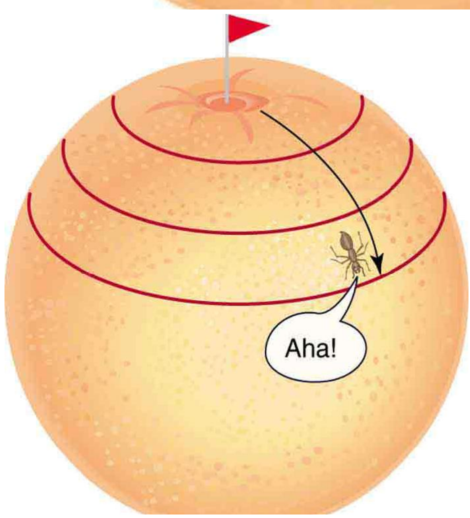
The surface is finite, but has no edge.

For a creature living on the sphere, having **no sense of the third dimension**, there is no center (on the sphere). All points are equal.

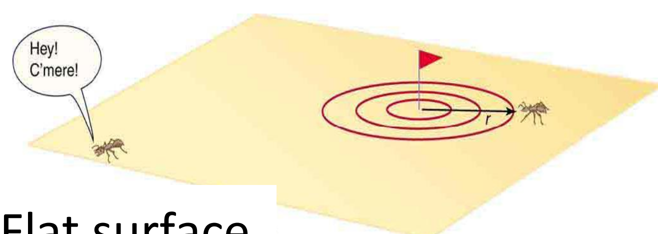
Any point on the surface can be defined as the center of a coordinate system.

But, how can a 2-D creature investigate the geometry of the sphere?

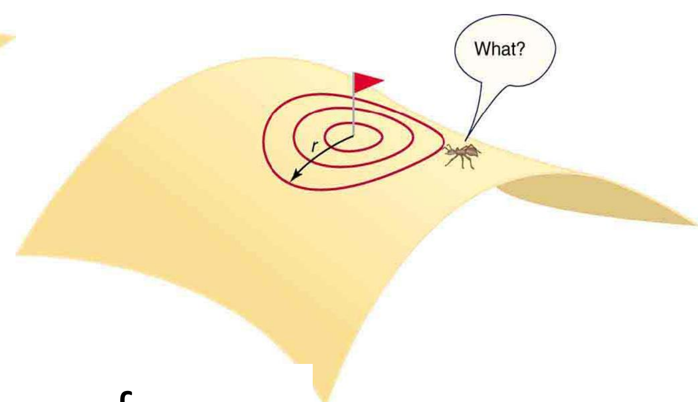
Answer: **Measure curvature of its space.**



Closed surface
(positive curvature, $k = 1$)



Flat surface
(zero curvature, $k = 0$)



Open surface
(negative curvature, $k = -1$)

Geometry of the Universe

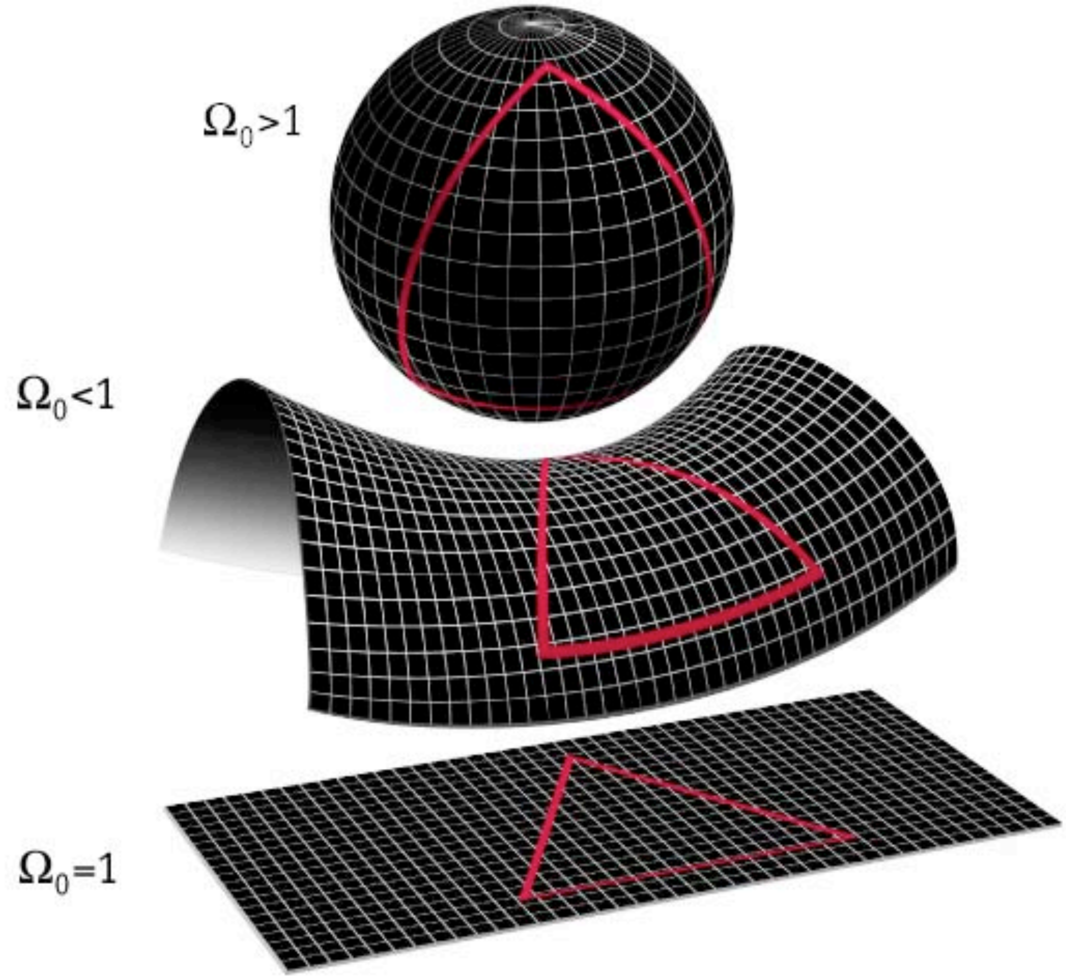
These are the three possible geometries of the Universe:

closed, **open** and **flat**, corresponding to a density parameter $\Omega_m = \rho/\rho_{crit}$ which is greater than, less than or equal to 1.

The relation to the curvature parameter is given by Eq. (3.36) (with $\Lambda = 0$).

The closed universe is of finite size. Traveling far enough in one direction will lead back to one's starting point.

The open and flat universes are infinite and traveling in a constant direction will never lead to the same point



3.6 - Static Universe

The static Universe requires $a = a_0$ = constant and thus $d^2a/dt^2 = da/dt = 0$. From Eq. (3.22), $d^2a/dt^2 = 0$ requires that

$$\Lambda = 4\pi G(\rho + 3p) = 4\pi G(1+\gamma)\rho \qquad (3.40)$$

If there is no cosmological constant ($\Lambda = 0$) then either $\rho = 0$ which is an empty Universe, or $p = -\rho/3$ which requires negative pressure. Both of these alternatives were unacceptable to Einstein and therefore he concluded that a cosmological constant was present, i.e. $\Lambda \neq 0$. From Equation (3.40) this implies

$$\rho = \frac{\Lambda}{4\pi G(1+\gamma)} \qquad (3.41)$$

and because ρ is positive this requires a positive Λ. Inserting Eq. (3.41) into Eq. (3.25), it follows that

$$\Lambda = \frac{3(1+\gamma)}{3+\gamma}\left[\left(\frac{\dot{a}}{a_0}\right)^2 + \frac{k}{a_0^2}\right] \qquad (3.42)$$

Now imposing $da/dt = 0$ and assuming a matter equation of state ($\gamma = 0$) implies $\Lambda = k/a_0^2$. However the requirement that Λ be positive forces $k = +1$, giving

$$\boxed{\Lambda = \frac{1}{a_0^2} = \text{constant}} \qquad (3.43)$$

Static Universe

Thus the cosmological constant for the static Universe is not any value but rather simply the inverse of the scale factor squared, where the scale factor has a fixed value in this static model.

Using Eq. (3.40), we obtain that the static Universe is closed with the scale factor (which in this case gives the radius of curvature) given by (**Einstein radius**)

$$a_0 = \frac{1}{\sqrt{4\pi G\rho_0}} \quad (3.44)$$

Using $\rho_0 = \rho_{crit}$ the numerical value of Einstein radius is of order of 10^{10} light years.

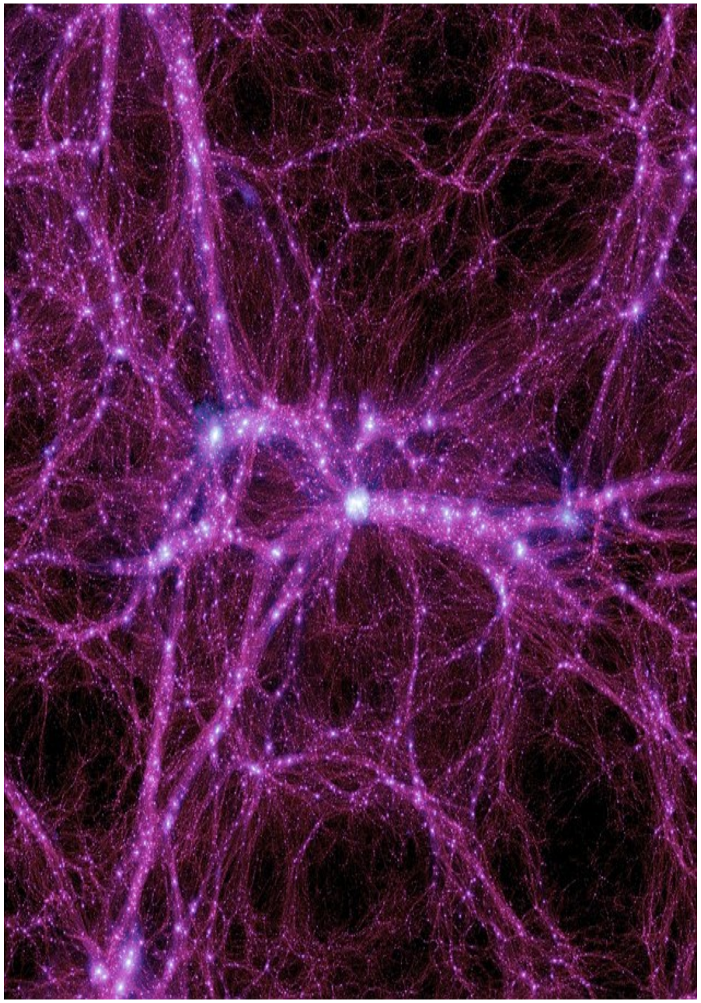
It is worth noting that even though the model is static, it is unstable:

if perturbed away from the equilibrium radius, the Universe will either expand to infinity or collapse. If we increase a from a_0, then the Λ term will dominate the equations, causing a runaway expansion, whereas if we decrease a from a_0, the dust (matter) term will dominate, causing collapse. Therefore, this model is also physically unsound, and this is a far worse problem than the (to Einstein) unattractive presence of Λ.

3.7 - Matter and Radiation Universes

Equation (3.5) can be rewritten as

$$\dot{\rho} + 3(p+\rho)\frac{\dot{a}}{a} = 0 \qquad (3.45)$$



or

$$\frac{d}{dt}\left(\rho a^3\right) + p\frac{da^3}{dt} = 0 \qquad (3.46)$$

From Eq. (3.13), $p = \gamma\rho/3$, from which follows that

$$\frac{d}{dt}\left(\rho a^{3+\gamma}\right) = 0 \qquad (3.47)$$

Integrating this we obtain

$$\boxed{\rho = \frac{c}{a^{3+\gamma}}} \qquad (3.48)$$

where c is a constant. This shows that the density falls as a^{-3} for matter dominated and a^{-4} for radiation-dominated Universes.

Matter and Radiation Universes

We consider here a flat, $k = 0$, Universe. Currently the Universe is in a matter dominated phase whereby the dominant contribution to the energy density is due to matter. However the early Universe was radiation dominated and the very early Universe was vacuum dominated. With $k = 0$, there will only be one term on the right hand side of Eq. (3.27) depending on what is dominating the Universe.

For a matter ($\gamma = 0$) or radiation ($\gamma = 1$) dominated Universe the right hand side of Eq. (3.27) will be of the form $1/a^{3+\gamma}$ (ignoring the vacuum energy), whereas for a vacuum dominated Universe the right hand side will be a constant. The solution to the Friedmann equation for a **radiation dominated Universe** will thus be (from $a\,da \propto dt$)

$$\boxed{a \approx t^{1/2}} \qquad (3.49)$$

while for the **matter dominated** case it will be (from $a^{1/2}da \propto dt$)

$$\boxed{a \approx t^{2/3}} \qquad (3.50)$$

One can see from d^2a/dt^2 that these results give negative acceleration, corresponding to a decelerating expanding Universe.

Summary: Solutions of Friedmann Equation

Friedmann equation for $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\rho \qquad (3.51)$$

To solve one needs $\rho(a)$. There are two important cases:

- $\rho \sim 1/a^4$ (**radiation-dominated Universe**). Then

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{1/2} \; ; \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \qquad (3.52)$$

- $\rho \sim 1/a^3$ (**matter-dominated Universe**). Then

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{2/3} \; ; \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \qquad (3.53)$$

Here the index '0' refers to the values today. As usual, we have set $a(t_0) = 1$.

Decceleration parameter

Deceleration parameter of the Universe (definition):

$$q = -\frac{\ddot{a}(t)}{H^2 a(t)} \qquad (3.54)$$

If the Universe is matter dominated, i.e., $p = 0$, then Eq. (3.27) (with $\rho_{vac} = 0$) yields

$$\rho = \frac{3H^2}{4\pi G} q \qquad (3.55)$$

Plugging this result into the Friedmann Equation (3.17), one gets

$$-k = a^2 H^2 (1 - 2q) \qquad (3.56)$$

Since both d$a/dt = 0$ and $H = 0$, for **flat Universe** ($k = 0$) we get $q = 1/2$. When combined with Eq. (3.55), this yields the critical density, Eq. (3.32), the density needed to yield the flat Universe. We also get $q > 1/2$ if $k = 1$ and $q < 1/2$ if $k = -1$.

The quantity q provides the relationship between the density of the Universe and the critical density,

$$q = \frac{\rho}{2\rho_{crit}} \qquad (3.57)$$