4- Cosmology - II

4.1 - Solutions of Friedmann Equation

As shown in Lecture 3, **Friedmann equation** is given by

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \qquad (4.1)$$

Using $\Lambda = 8\pi G \rho_{vac}$ it can also be written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho + \rho_{vac}\right) - \frac{k}{a^2} \qquad (4.2)$$

Using Newton's 2[nd] law, the Gravitational law, and $p = \gamma\rho/3$ for matter $\gamma = 0$ or radiation $\gamma = 1$, one can also rewrite it as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(1+\gamma\right)\rho + \frac{8\pi G}{3}\rho_{vac} \qquad (4.3)$$

In this lecture, we discuss the applications of these equations to the **structure** and **dynamics** of the **Universe**.

4.2 - Flat Universe

In this case, the Universe has no curvature, *k = 0* (and using Λ = *0*), and Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad (4.4)$$

As we saw in Eq. (3.56) *q = 1/2*. We consider the matter-dominated and the radiation-dominated Universes separately.

4.2.1 - Matter dominated Universe

We have ***p = 0*** and $a^3\rho$ ***= const*** and, using Eq. (4.4) (with $\Lambda = 0$), we get

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 \qquad (4.5)$$

which leads to

$$\int a^{1/2}\,da = \frac{2}{3}a^{3/2} + C = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\;t \qquad (4.6)$$

Flat, matter dominated Universe

At the Big Bang, $t = 0$, $a = 0$, so $C = 0$. Since the Universe is assumed flat, $k = 0$, $\rho_0 = \rho_{crit}$, we have

$$\frac{a}{a_0} = \left(6\pi G\rho_0\right)^{1/3} t^{2/3} = \left(6\pi G\rho_{crit}\right)^{1/3} t^{2/3} = \left(6\pi G\frac{3H_0^2}{8\pi G}\right)^{1/3} t^{2/3}$$

or

$$\frac{a}{a_0} = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3} \qquad (4.7)$$

From this we compute the age of the Universe t_0, which corresponds to the Hubble rate H_0 and the scale factor $a = a_0$, to be

$$t_0 \approx 9.1\times 10^9 \text{ years} \equiv 9.1 \text{ aeon} \qquad (4.8)$$

Notice that we had already obtained the correct power dependence of a as a function of t in Eq. (3.53). Now we also obtained the correct multiplicative coefficient in Eq. (4.7).

The model described in this subsection in known as the **Einstein–de Sitter model**.

4.2.1 - Flat, Radiation dominated Universe

We have $p = \rho/3$ and $a^4\rho = const$ and, using Eq. (4.2) (with $\Lambda = 0$), we get

$$\int a\,da = 2a^2 + C = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\,t \qquad (4.9)$$

Again, at the Big Bang, $t = 0$, $a = 0$, thus $C = 0$, and ρ_0 is equal to the critical density ρ_{crit}. Therefore,

$$\frac{a}{a_0} = \left(\frac{2\pi}{3}G\rho_{crit}\right)^{1/4} t^{1/2} = \left(\frac{H_0}{2}\right)^{1/2} t^{1/2} \qquad (4.10)$$

For a radiation-dominated Universe, the age of the Universe would be much longer than for a matter-dominated Universe:

$$t_0 \approx 27 \text{ aeon} \qquad (4.11)$$

Notice that we had already obtained the correct power dependence of a as a function of t in Eq. (3.52). Now we also obtained the correct multiplicative coefficient in Eq. (4.10).

4.3 - Closed Universe

In this case, we have $k = 1$ and $q > \frac{1}{2}$. Here we only present the results.

4.3.1 - Matter dominated

We have $p = 0$ and $a^3\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \frac{q}{2q-1}(1-\cos\eta), \qquad \text{with} \qquad \frac{t}{t_0} = \frac{q}{2q-1}(\eta - \sin\eta) \tag{4.12}$$

where q is the decceleration parameter, given by Eq. (3.54).

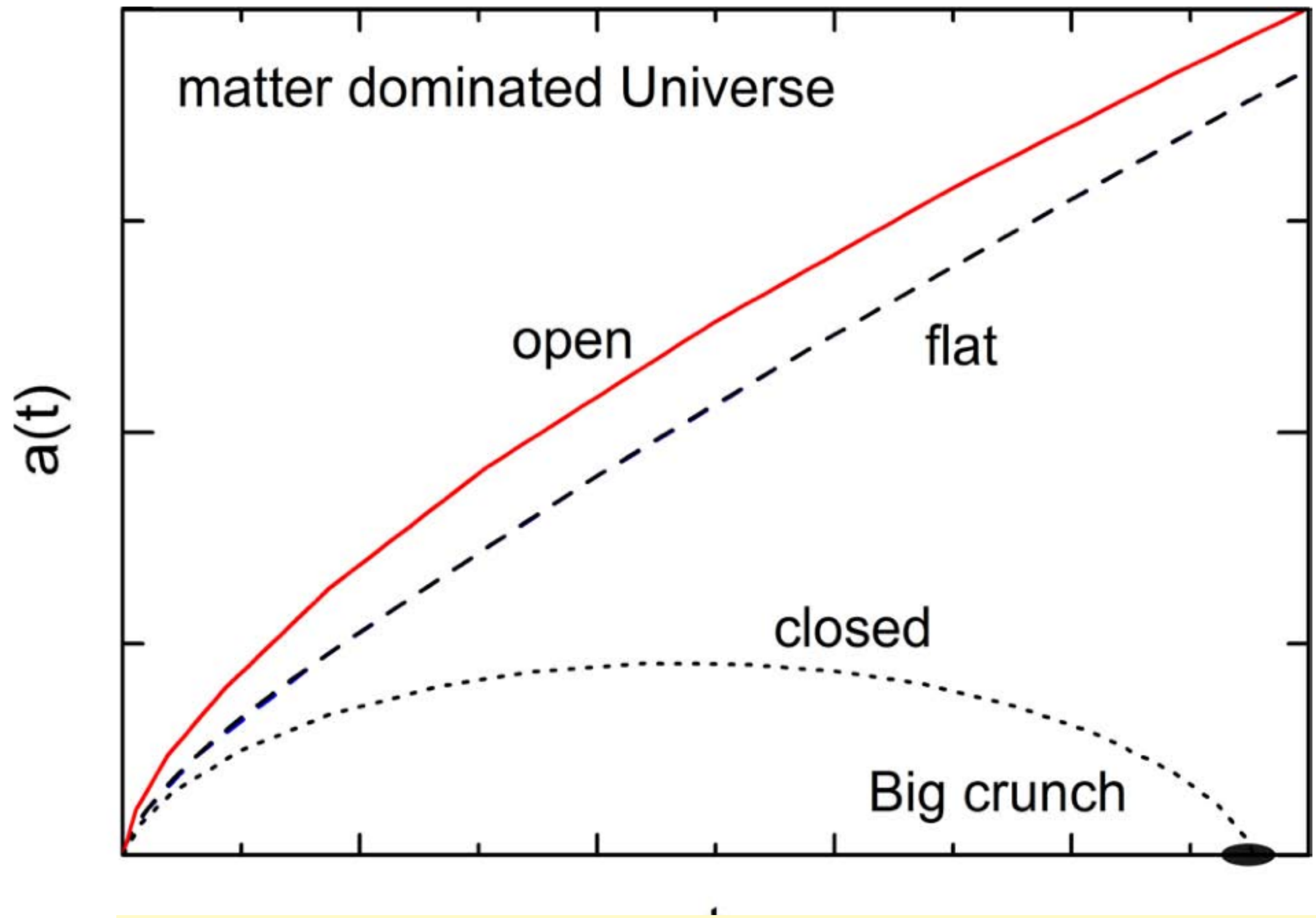
4.3.2 - Radiation dominated

We have $p = \rho/3$ and $a^4\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \sqrt{\frac{2q}{2q-1}}\sin\eta, \qquad \text{with} \qquad \frac{t}{t_0} = \sqrt{\frac{2q}{2q-1}}(1-\cos\eta) \tag{4.13}$$

In both matter- and radiation-dominated closed Universes, the evolution is cycloidal — the scale factor grows at an ever-decreasing rate until it reaches a point at which the expansion is halted and reversed. The Universe then starts to compress and it finally collapses in the **Big Crunch**. (see next figure.)

Matter Dominated Universe



Evolution of the scale factor *a(t)* for the flat, closed and open matter-dominated Friedmann Universe.

4.4 - Open Universe

In this case, we have $k = -1$ and $q < \frac{1}{2}$. Here we only present the results.

4.4.1 - Matter dominated

We have $p = 0$ and $a^3\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \frac{q}{2q-1}(\cosh\eta - 1), \quad \text{with} \quad \frac{t}{t_0} = \frac{q}{2q-1}(\sinh\eta - 1) \tag{4.14}$$

4.4.2 - Radiation dominated

We have $p = \rho/3$ and $a^4\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \sqrt{\frac{2q}{1-2q}}\sinh\eta, \quad \text{with} \quad \frac{t}{t_0} = \sqrt{\frac{2q}{1-2q}}(\cosh\eta - 1) \tag{4.15}$$

The previous figure summarizes the evolution of the scale factor a(t) for open, flat and closed matter-dominated Universes. In earlier times, one can expand the trigonometric and hyperbolic functions to leading terms in powers of η, and the a and *t* dependence on η for the different curvatures are given in the next table. This shows that at early times the curvature of the Universe does not matter. The singular behavior at early times is essentially independent of the curvature of the Universe, or k. **The Big Bang is a matter dominated singularity.**

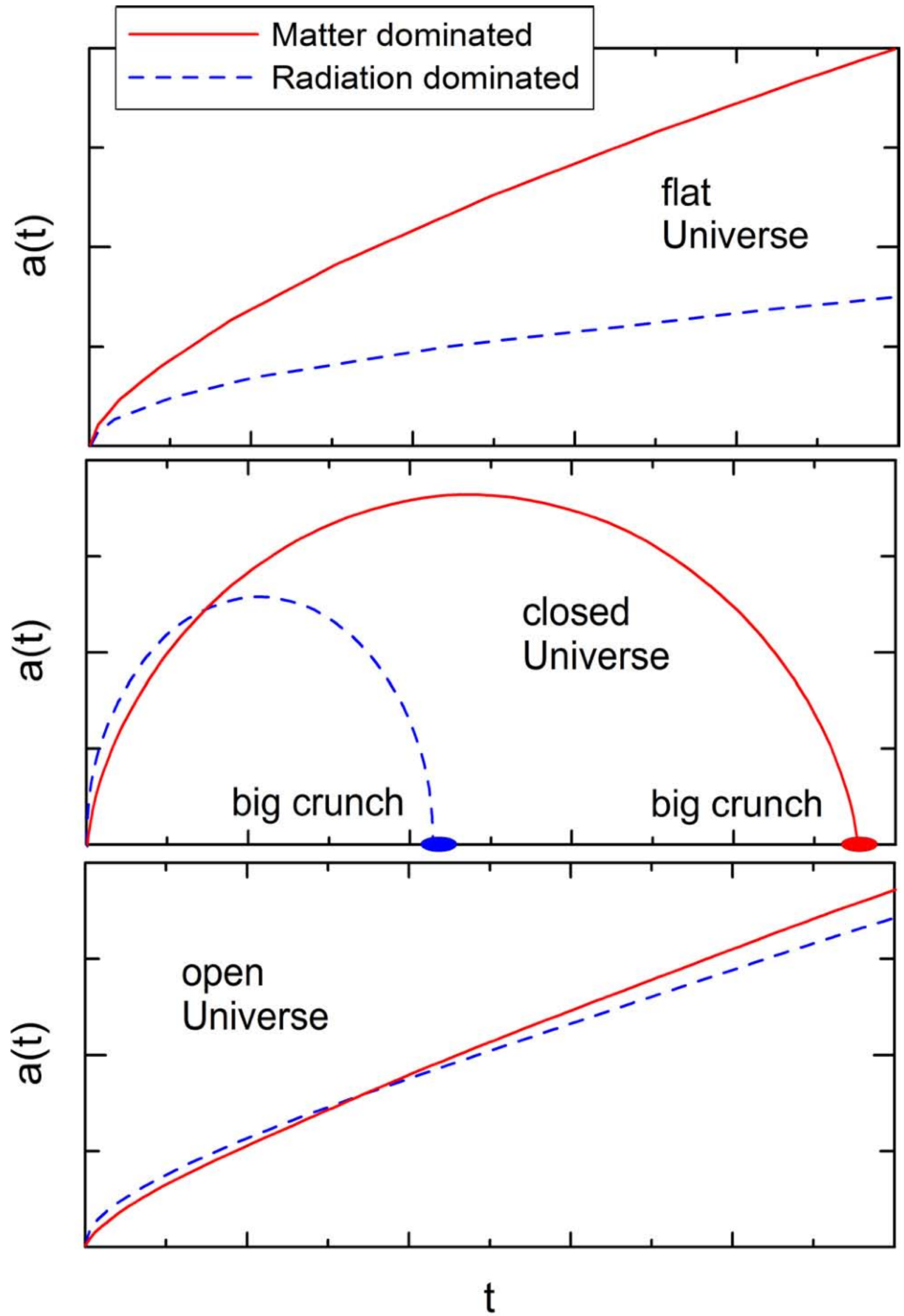
Matter Dominated Universe

Curvature	All η		Small η		
k	a/a_0	t	a	t	$a(t)$
0	$(6\pi G\rho_0)^{1/3}t^{2/3}$	—	$\propto t^{2/3}$	—	$\propto t^{2/3}$
1	$\frac{q}{2q-1}(1-\cos\eta)$	$\frac{q}{2q-1}(\eta-\sin\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$
-1	$\frac{q}{1-2q}(\cosh\,\eta-1)$	$\frac{q}{1-2q}(\sinh\,\eta-\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$

Solutions of Friedmann equation for matter-dominated Universe.

Summary: Solutions of Friedmann Equation

Evolution of the scale factor a(t) for the flat (upper panel), closed (middle panel) and open (lower panel) Friedmann Universe.



Flatness problem

According to Eq. (3.36), the density parameter satisfies an equation of the form

$$|\Omega - 1| = \frac{|\rho(t) - \rho_c|}{\rho_c} = \frac{1}{\dot{a}^2(t)} \qquad (4.16)$$

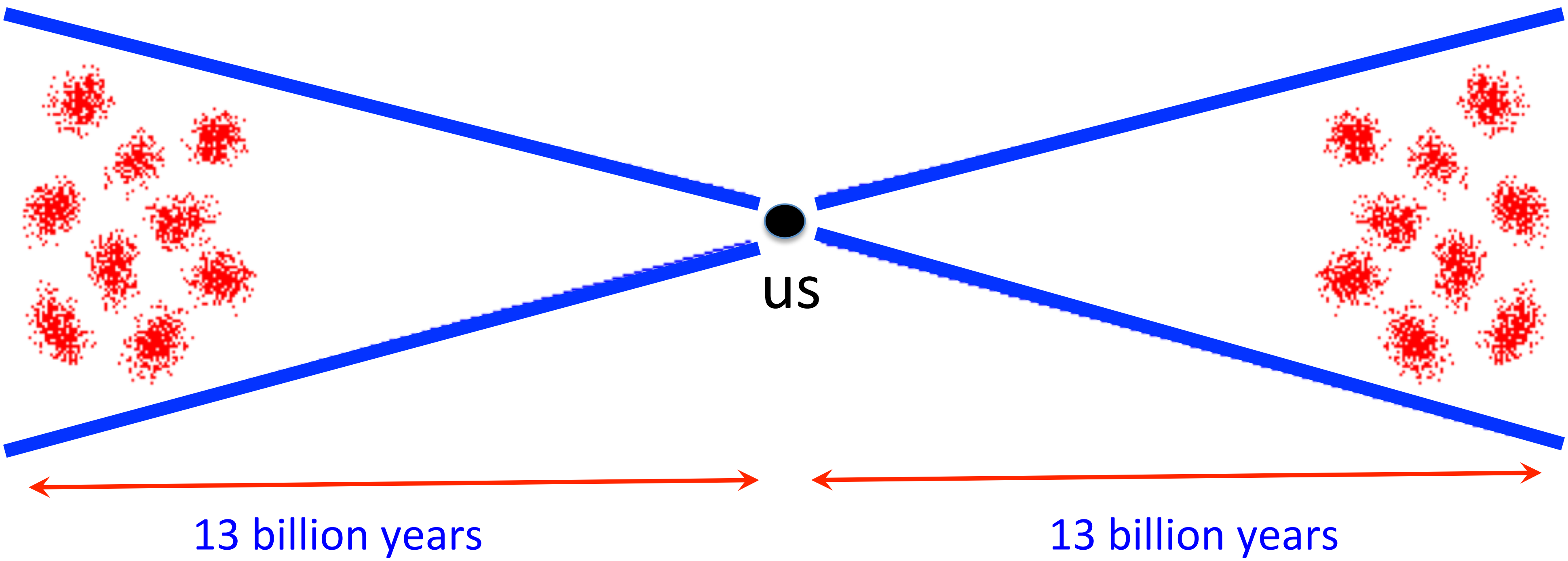
The present day value of Ω is known only roughly, $0.1 \leq \Omega \leq 2$. On the other hand $1/(da/dt)^2 \sim 1/t^2$ in the early stages of the evolution of the Universe, so the quantity $|\Omega - 1|$ was extremely small. One can show that in order for Ω to lie in the range $0.1 \leq \Omega \leq 2$ now, the early Universe must have had

$$|\Omega - 1| = \frac{|\rho(t) - \rho_c|}{\rho_c} \leq 10^{-59} \qquad (4.17)$$

This means that if the density of the Universe were initially greater than ρ_c, say by $10^{-55}\rho_c$, it would be closed and the Universe would have collapsed long time ago. If on the other hand the initial density were $10^{-55}\rho_c$ less than ρ_c, the present energy density in the Universe would be vanishingly low and the life could not exist. The question of why the energy density in the early Universe was so fantastically close to the critical density is usually known as the **flatness problem**.

Horizon problem

The number and size of density fluctuations on both sides of the sky are similar, yet they are separated by a distance that is greater than the speed of light times the age of the Universe, i.e. they should have no knowledge of each other by special relativity (c, the speed of light is finite: c = 300,000 km/s).



At some time in the early Universe, all parts of spacetime were causally connected, this must have happened after the spacetime foam era, and before the time where thermalization of matter occurred. This is known as the **horizon problem**.

Inflation

Inflation occurs when the vacuum energy contribution dominates the ordinary density and curvature terms in Friedmann Equation (4.1). Assuming these are negligible and substituting Λ = constant, one gets

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = +\frac{\Lambda}{3} \qquad (4.18)$$

The solution is

$$a(t) = \exp\left\{\sqrt{\frac{\Lambda}{3}}t\right\} \qquad (4.19)$$

When the Universe is dominated by a cosmological constant, the **expansion rate grows exponentially**.

Inflation phase

Particle physics predicts that the interactions among particles in the early Universe are controlled by forces (or force fields (sf)), which can have negative pressure ($p_{sf} = -\rho_{sf}$, where ρ_{sf} is the energy density of the fields.)
If $\rho_{sf} \gg \rho_{rad}$, Friedmann equation reads:

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left[\rho_{sf} + 3p_{sf}\right]a = \frac{8\pi G}{3}\rho_{sf}a \qquad (4.20)$$

The solution is:

$$a(t) = a_0 \exp\left\{\frac{t}{t_{\text{int}}}\right\}; \qquad t_{\text{int}} = \sqrt{\frac{3}{8\pi G\rho_{sf}}} \qquad (4.21)$$

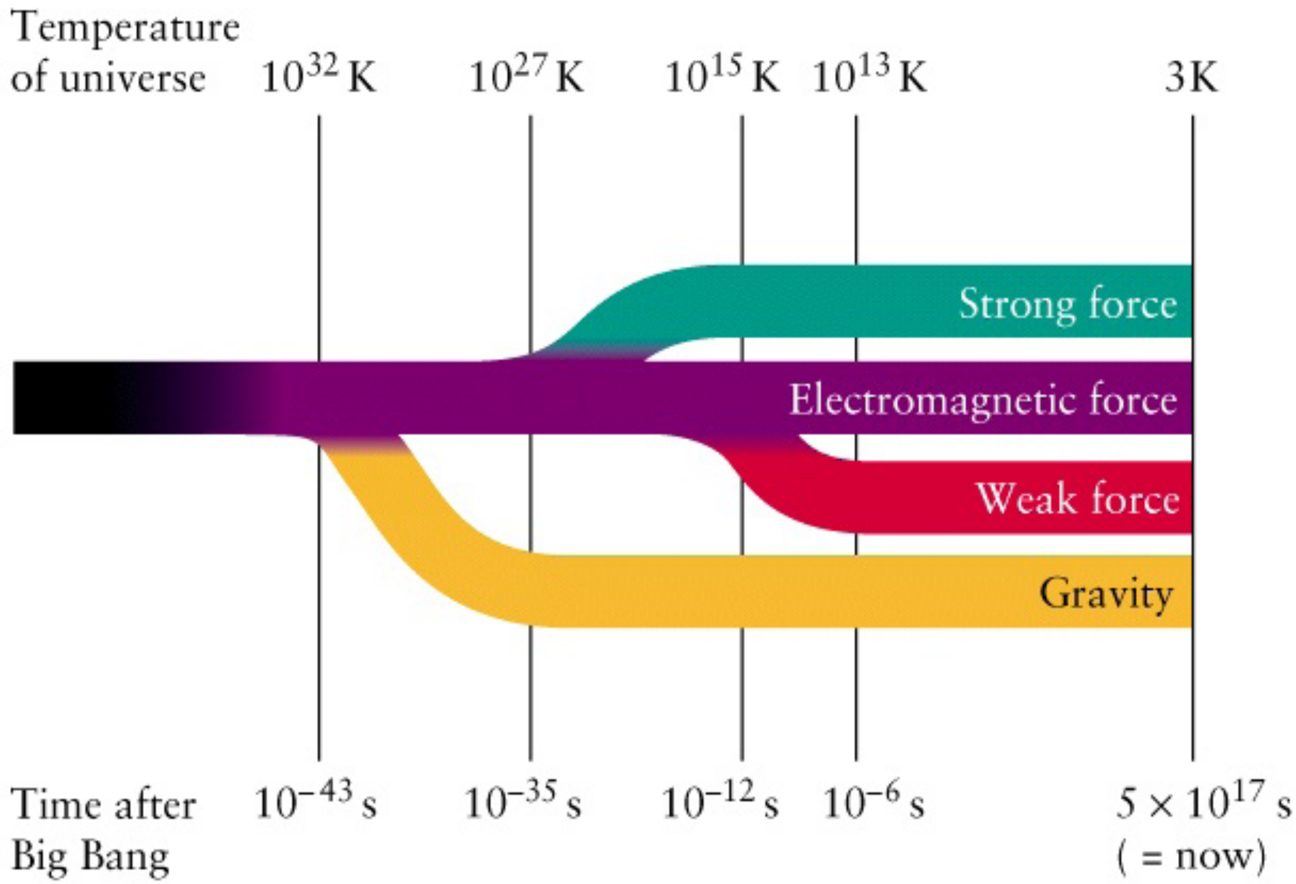
The model predicts $t_{inf} \sim 10^{-34}$ s. Inflation lasts as long as the energy density associated with the force fields is transformed into expansion of the Universe. If this lasts from $t_1 = 10^{-34}$ s to $t_2 = 10^{-32}$ s, then the Universe expands by a factor $exp(\Delta t/t_{inf}) \sim e^{100} \sim 10^{43}$.

After this exponential increase, the Universe develops as we discussed before.

Grand Unification Theory (GUT) of forces

The separation of forces happens in phase transitions with spontaneous symmetry breaking.

NOTE: we will discuss more about forces and particles in a forthcoming lecture.



kT (temperature)	united forces	big bang time
~ 10^2 GeV	electromagnetic + weak	
~ 10^{13} GeV	electromagnetic + weak + strong	~ 10^{-35} s
~ 10^{19} GeV	electromagnetic + weak + strong + gravity	~ 10^{-43} s Planck time

Inflation

There are a number of other observations that are suggestive of the need for a **cosmological constant**. For example, if the cosmological constant today comprises most of the energy density of the universe, then the extrapolated age of the universe is much larger than it would be without such a term, which helps avoid the dilemma that the extrapolated age of the universe is younger than some of the oldest stars we observe!

A cosmological constant term added to the inflationary model, an extension of the Big Bang theory, leads to a model that appears to be consistent with the observed large-scale distribution of galaxies and clusters, with COBE's measurements of cosmic microwave background fluctuations, and with the observed properties of X-ray clusters.

The **Inflation Theory** proposes a period of extremely rapid (exponential) expansion of the universe shortly after the Big Bang.

While the Big Bang theory successfully explains the shape of the cosmic microwave background spectrum and the origin of the light elements, it leaves open a number of important questions:

Why is the universe so uniform on the largest length scales?

Why is the physical scale of the universe so much larger than the fundamental scale of gravity, the Planck length, which is one billionth of one trillionth of the size of an atomic nucleus? Why are there so many photons in the universe? What physical process produced the initial fluctuations in the density of matter?

The Inflation Theory, developed by Alan Guth, Andrei Linde, Paul Steinhardt, and Andy Albrecht, offers answers to these questions and several other open questions in cosmology. It proposes a period of extremely rapid (exponential) expansion of the universe leading to the Big Bang expansion, during which time the energy density of the universe was dominated by a cosmological constant term that later decayed to produce the matter and radiation that fill the universe today. The Inflation Theory links important ideas in modern physics, such as symmetry breaking and phase transitions, to cosmology.

The Inflation Theory makes a number of important predictions: (a) the density of the universe is close to the critical density, and thus the geometry of the universe is flat. (b) the fluctuations in the primordial density in the early universe had the same amplitude on all physical scales. (c) there should be, on average, equal numbers of hot and cold spots in the fluctuations of the cosmic microwave background temperature.

Planck mass, time and length

Before t = 10^{-3} s, the Universe was dominated by quantum mechanics.
A measure for this point in time is the Compton wavelength of the Universe

$$R_C = \frac{\hbar}{Mc} \qquad (4.23)$$

where M is all the mass in the Universe, ħ = h/2π, with h = 4.136 × 10^{-21} MeV.s. the **Planck constant.**

The **Schwarzschild radius** is defined as the boundary between spacetime regions around an object of mass M that can and cannot communicate with the external Universe. It is given by

$$R_S = \frac{2GM}{c^2} \qquad (4.24)$$

Equating the Compton wavelength and the Schwartzschild radius leads to the so-called **Planck mass**,

$$M_{Pl} = \left(\frac{\hbar c}{G}\right)^{1/2} = 1.2 \times 10^{19} \quad \mathrm{GeV}/c^2 \qquad (4.25)$$

which is equivalent to a temperature T = 10^{32} K, using $M_{Pl}c^2 \sim kT$.

Via the uncertainty relation from quantum mechanics, $\Delta E \Delta t \sim \hbar/2$, or $\Delta x \Delta p \sim \hbar/2$, one can get a **Planck time and length**, i.e. t_{Pl} = 5 × 10^{-44} s and l_{Pl} = 3 × 10^{-33} cm. At these lengths and times, time itself becomes undefined.

In typical models of inflation the phase transition takes place at temperatures around the Grand Unification Temperature (GUT) when the Universe was about 10^{-34} s old. Suppose that the Universe stayed in the inflationary state for 10^{-32} s. This may appear to be a short time, but in fact inflation lasted for 100 hundred times the age of the Universe at the time inflation started.

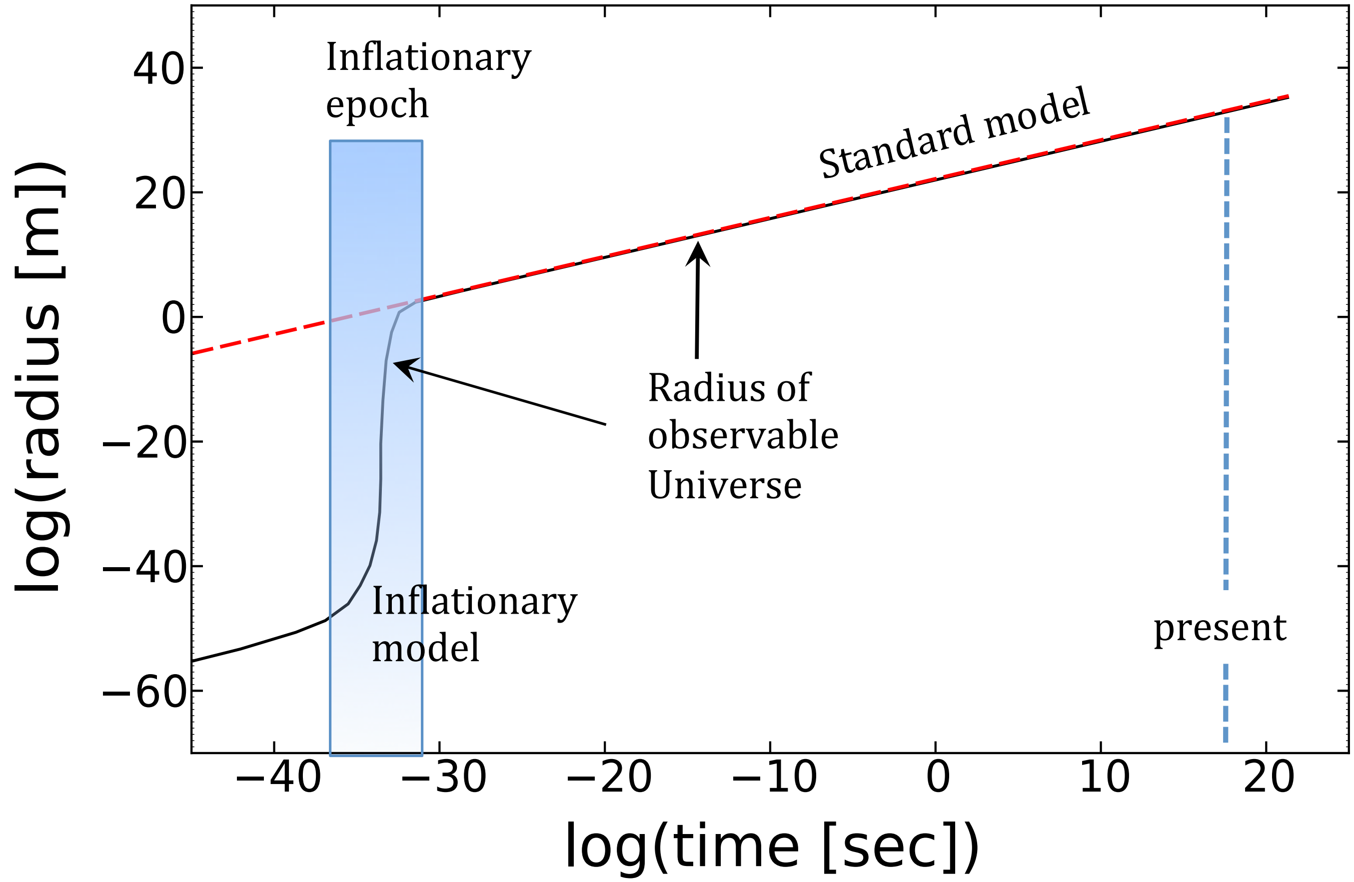
Consider a small region with radius around, say, 10^{-23} cm before inflation. After inflation the volume of the region has increased by a factor of $(e^{100})^3$ = 1.9 × 10^{130}. This huge increase solves some of the problems previously mentioned: (a) flatness and (b) horizon problems.

Inflation solves the horizon problem

The CMB is found to very smooth ($\Delta T/T \sim 10^{-5}$). This suggests that the different parts of our universe communicated before they equilibrated.
Light can only travel a finite distance during the finite age of Universe and photons of the CMB, coming from opposite directions, and observed today can have no information from each other. However, inflation allows every part of the Universe to have been in close proximity before the inflation phase.

After the end of inflation the vacuum energy of the field driving inflation was transferred to ordinary particles, so a reheating of the Universe took place. During inflation the Universe itself supercooled with $T \sim e^{-Ht}$. The period of inflation and reheating is strongly non-adiabatic, since there is an enormous generation of entropy at reheating. After the end of inflation, the Universe restarts in an adiabatic phase with the standard conservation of the energy. In fact the Universe restarts from very special initial conditions that the horizon and flatness problems are avoided.

Inflation solves the flatness problem



In the cosmological models the k/a^2 term, with $k = \pm 1$, could have been important. However, with the sudden inflation the scale factor has increased by ~ 10^{29} and the value of k/a^2 has been reduced by 10^{58}. So, even if we started from a large curvature term there is no other mechanism necessary to arrive at the nearly flat Universe.

During inflation the energy density of the Universe is constant, whereas the scale factor increases exponentially, as described by Eqs. (4.19) of (4.21). This means that Ω must have been exponentially close to unity, $\Omega = 1$ to an accuracy of many decimal places.