5 - Stellar Structure I

5.1 - The Equations of Stellar Structure

What are the main physical processes which determine the structure of stars ?

- Stars are held together by gravitation – attraction exerted on each part of the star by all other parts
- Collapse is resisted by internal thermal pressure.
- These two forces play the principal role in determining stellar structure – they must be (at least almost) in balance.
- Thermal properties of stars – continually radiating into space. If thermal properties are constant, continual energy source must exist.
- Theory must describe - origin of energy and transport to surface.

We make two fundamental assumptions :

1) Neglect the rate of change of properties – assume constant with time

2) All stars are spherical and symmetric about their centers

The Equations of Stellar Structure

For our stars – which are isolated, static, and spherically symmetric – there are four basic equations to describe structure. All physical quantities depend on the distance from the center of the star alone

1) **Equation of hydrostatic equilibrium:** at each radius, forces due to pressure differences balance gravity

2) **Conservation of mass**

3) **Conservation of energy** : at each radius, the change in the energy flux = local rate of energy release

4) **Equation of energy transport** : relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state - pressure of a gas as a function of its density and temperature

- Opacity - how opaque the gas is to the radiation field

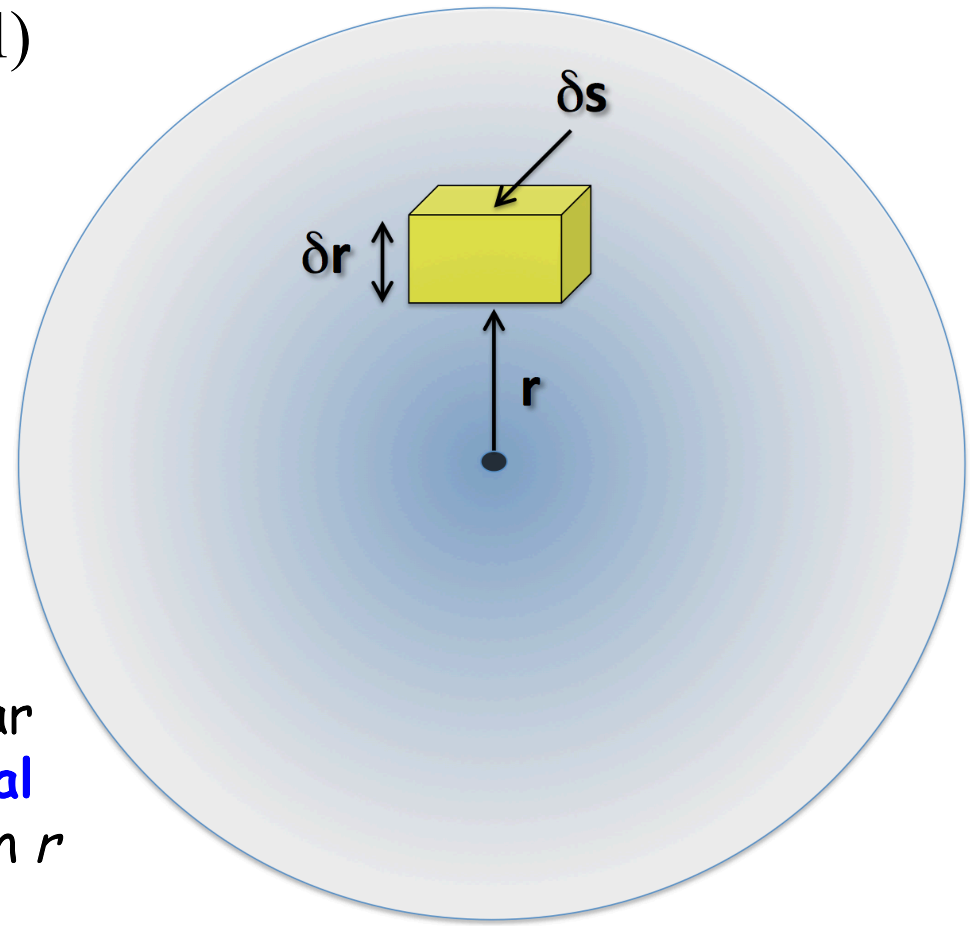
- Nuclear energy generation rate

5.1.1 - Equation of hydrostatic support

Balance between gravity and internal pressure is known as *hydrostatic equilibrium*

Mass of element
$$\delta m = \rho(r)\delta s \delta r \quad (5.1)$$

where $\rho(r)$ = density at r



Consider forces acting in radial direction

1. Outward force: **pressure** exerted by stellar material on the lower face:

$$P(r)\delta s \quad (5.2)$$

2. Inward force: pressure exerted by stellar material on the upper face, and **gravitational attraction** of all stellar material lying within *r*

$$P(r+\delta r)\delta s + \frac{GM(r)}{r^2}\delta m$$
$$= P(r+\delta r)\delta s + \frac{GM(r)}{r^2}\rho(r)\delta s \delta r \quad (5.3)$$

Equation of hydrostatic support

In hydrostatic equilibrium:

$$P(r)\delta s = P(r+\delta r)\delta s + \frac{GM(r)}{r^2}\rho(r)\delta s\,\delta r$$
$$\Rightarrow P(r+\delta r) - P(r) = -\frac{GM(r)}{r^2}\rho(r)\delta r \qquad (5.4)$$

If we consider an infinitesimal element, we write

$$\frac{P(r+\delta r) - P(r)}{\delta r} = \frac{dP(r)}{dr} \qquad (5.5)$$

Hence rearranging above we get

$$\boxed{\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}} \qquad (5.6)$$

the **equation of hydrostatic support**.

5.1.2 - Equation of mass conservation

Mass $M(r)$ contained within a star of radius r is determined by the **density of the gas** $\rho(r)$.

Consider a thin shell inside the star with radius r and outer radius $r+\delta r$

$$\delta V = 4\pi r^2 \delta r$$

$$\Rightarrow \delta M = \delta V \rho(r) = 4\pi r^2 \delta r \rho(r)$$

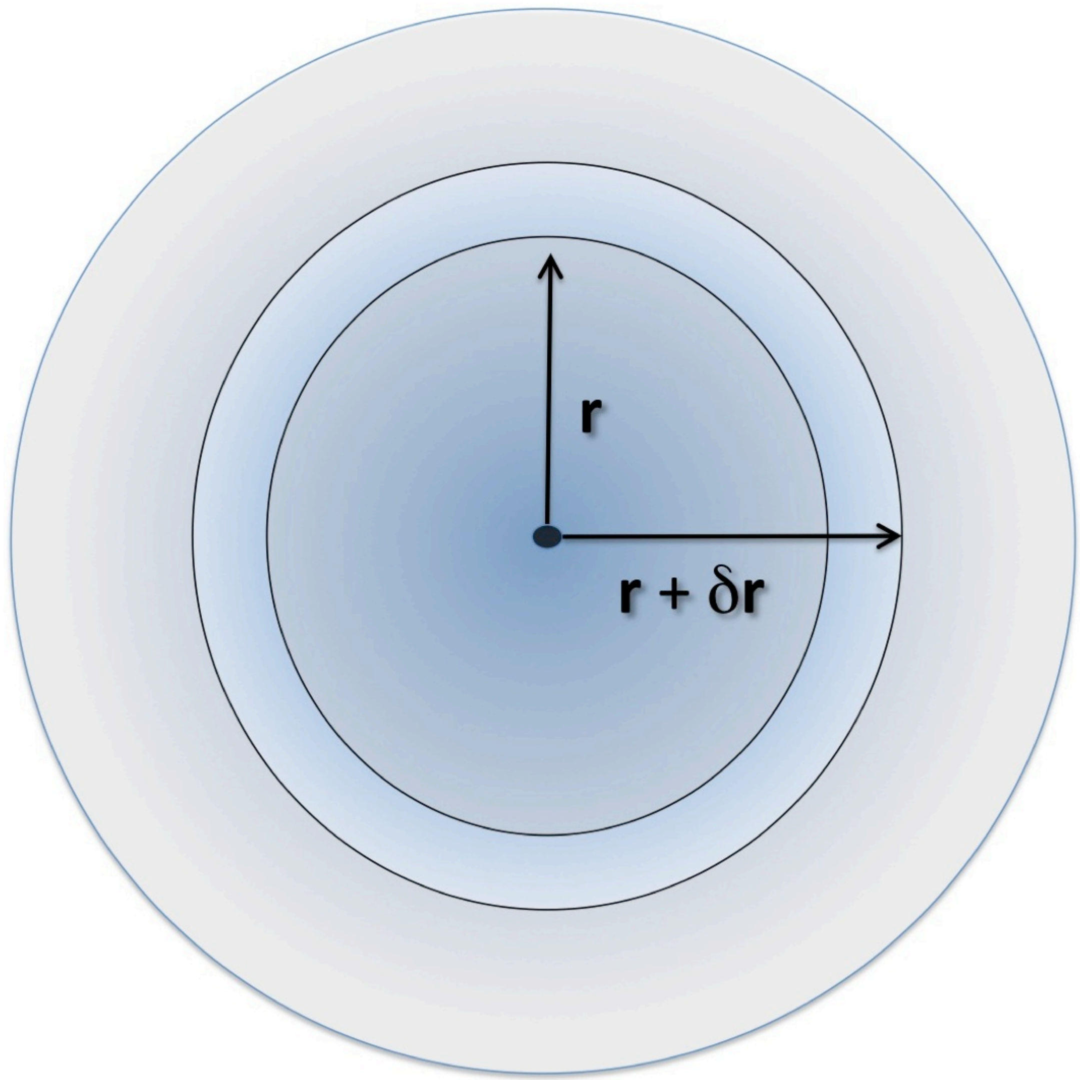
(5.7)

or

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)} \qquad (5.8)$$

In the limit where $\delta r \rightarrow 0$

this the **equation of mass conservation.**



5.1.3 - Accuracy of hydrostatic assumption

We have assumed that the gravity and pressure forces are balanced - how valid is that ?

Consider the case where the outward and inward forces are not equal, there will be a resultant force acting on the element which will give rise to an acceleration *a*

$$P(r+\delta r)\delta s + \frac{GM(r)}{r^2}\rho(r)\delta s\delta r - P(r)\delta s = \rho(r)\delta s\delta r a$$
$$\Rightarrow \frac{dP(r)}{dr} + \frac{GM(r)}{r^2}\rho(r) = \rho(r)a \qquad (5.9)$$

Now the local (at position r) acceleration due to gravity is $g = GM(r)/r^2$

$$\boxed{\frac{\mathrm{dP(r)}}{\mathrm{dr}} + \mathrm{g}\rho(\mathrm{r}) = \rho(\mathrm{r})\mathrm{a}} \qquad (5.10)$$

which is the **generalized form of the equation of hydrostatic support.**

Accuracy of hydrostatic assumption

Now suppose there is a resultant force on the element (LHS ≠ 0). Suppose their sum is small fraction of gravitational term (β)

$$\beta\rho(r)g = \rho(r)a \qquad (5.11)$$

Hence there is an inward acceleration of

$$a = \beta g \qquad (5.12)$$

Assuming it begins at rest, the spatial displacement *d* after a time *t* is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\beta g t^2 \qquad (5.13)$$

The dynamical timescale

If we allowed the star to collapse i.e. set ***d* = *r*** and substitute ***g* = *GM*/*r*²**

$$t = \frac{1}{\sqrt{\beta}}\left(\frac{2r^3}{GM}\right)^{\frac{1}{2}} \qquad (5.14)$$

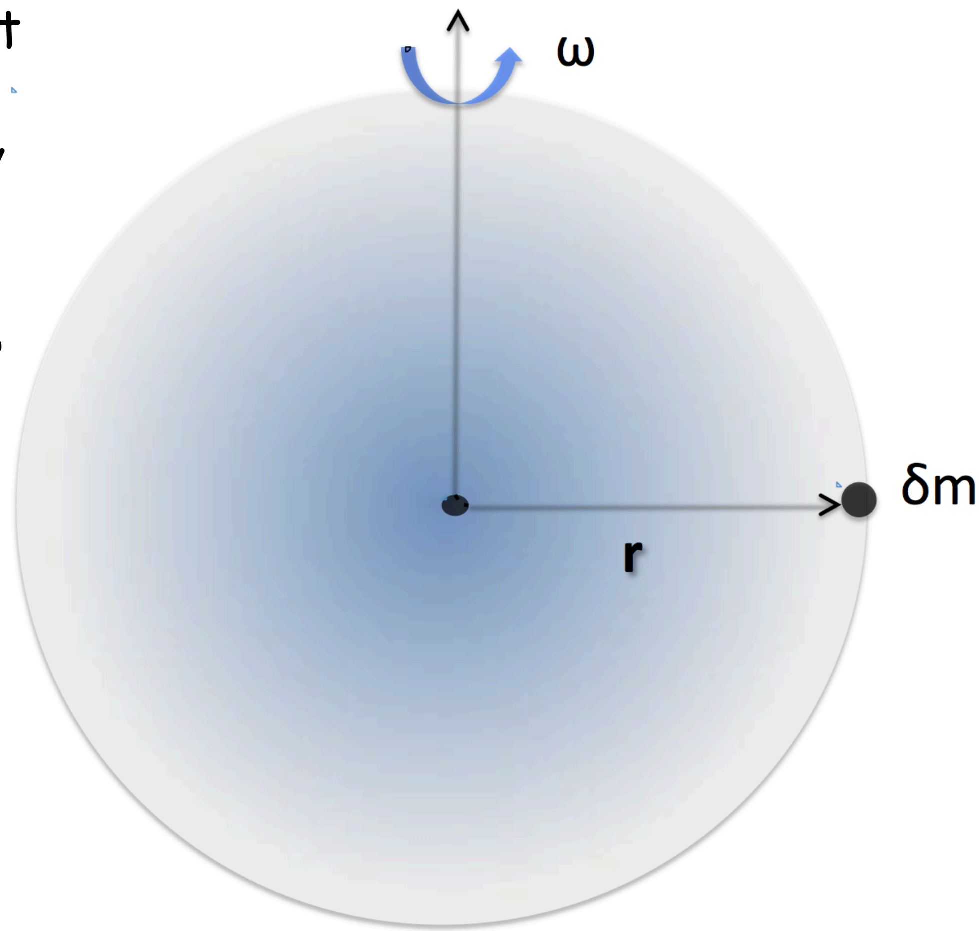
Assuming ***β* ~ 1**

$$t_d = \left(\frac{2r^3}{GM}\right)^{\frac{1}{2}} \qquad (5.15)$$

5.1.4 - Accuracy of spherical symmetry assumption

Stars are rotating gaseous bodies – to what extent are they flattened at the poles ? If so, departures from spherical symmetry must be accounted for

Consider mass ***δm*** near the surface of star of mass ***M*** and radius ***r***
Element will be acted on by additional inwardly acting force to provide circular motion.



Centripetal force is given by: $\delta m \omega^2 r$

Where ***ω = angular velocity of star***

There will be no departure from spherical symmetry provided that

$$\delta m \omega^2 r \Big/ \frac{GM\delta m}{r^2} << 1 \quad \text{or} \quad \omega^2 << \frac{GM}{r^3} \qquad (5.16)$$

Accuracy of spherical symmetry assumption

Note the RHS of last equation is similar to t_d

$$t_d = \left(\frac{2r^3}{GM}\right)^{\frac{1}{2}} \quad \text{or} \quad \frac{GM}{r^3} = \frac{2}{t_d^2}$$
$$\Rightarrow \omega^2 << \frac{2}{t_d^2} \qquad (5.17)$$

And as $\omega=2\pi/T$; *where* ***T = rotation period***,
if spherical symmetry is to hold then $T >> t_d$

For example **t_d (sun) ~ 2000 s** and **T ~ 1 month**

$\Rightarrow$ **For the majority of stars, departures from spherical symmetry can be ignored.**

Some stars do rotate rapidly and rotational effects must be included in the structure equations - can change the output of models

5.1.5 - Minimum value for central pressure of star

We have only 2 of the 4 equations, and no knowledge yet of material composition or physical state. But can deduce a minimum central pressure :

Why, in principle, do you think there needs to be a minimum value ? given what we know, what is this likely to depend upon ?

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

Divide these two equations:

$$\frac{dP(r)}{dr} \Big/ \frac{dM(r)}{dr} \equiv \frac{dP}{dM} = -\frac{GM}{4\pi r^4}$$

We can integrate this to give

$$P_c - P_s = \int_0^{M_s} \frac{GM}{4\pi r^4} dM \qquad (5.18)$$

Lower limit to RHS:

$$\int_0^{M_s} \frac{GM}{4\pi r^4} dM > \int_0^{M_s} \frac{GM}{4\pi r_s^4} dM = \frac{GM_S^2}{8\pi r_s^4} \qquad (5.19)$$

Minimum value for central pressure of star

Hence we have

$$P_c - P_s > \frac{GM_s^2}{8\pi r_s^4} \qquad (5.20)$$

We can approximate the pressure at the surface of the star to be zero:

$$\boxed{P_c > \frac{GM_s^2}{8\pi r_s^4}} \qquad (5.21)$$

For example for the Sun:

$P_{c\odot} = 4.5 \times 10^{13}$ Nm^{-2} = 4.5×10^{8} atmospheres

This seems rather large for gaseous material – we shall see that this is not an ordinary gas.

5.2 - The Virial theorem

Again lets take the two equations of hydrostatic equilibrium and mass conservation and divide them

$$\frac{dP(r)}{dr} \bigg/ \frac{dM(r)}{dr} \equiv \frac{dP}{dM} = -\frac{GM}{4\pi r^4} \qquad (5.22)$$

Now multiply both sides by $4\pi r^2$

$$4\pi r^3 dP = -\frac{GM}{r} dM \qquad (5.22)$$

And integrate over the whole star

$$3\int_{P_c}^{P_s} V dP = -\int_0^{M_s} \frac{GM}{r} dM \qquad (5.23)$$

Where V = vol contained within radius r

Use integration by parts to integrate LHS

$$3\left[PV\right]_c^s - 3\int_{V_c}^{V_s} P dV = -\int_0^{M_s} \frac{GM}{r} dM \qquad (5.24)$$

At center, $V_c = 0$ and at surface $P_s = 0$

The Virial theorem

Hence we have

$$3\int_0^{V_s} PdV - \int_0^{M_s} \frac{GM}{r} dM = 0 \qquad (5.25)$$

Now the right hand term = total gravitational potential energy of the star or it is the energy released in forming the star from its components dispersed to infinity.

Thus we can write the **Virial Theorem**:

$$3\int_0^{V_s} PdV + \Omega = 0 \qquad (5.26)$$

This is of great importance in astrophysics and has many applications. We shall see that it relates the gravitational energy of a star to its thermal energy.

5.2.1 - Minimum mean temperature of a star

We have seen that pressure, *P,* is an important term in the equation of hydrostatic equilibrium and the Virial theorem. We have derived a minimum value for the central pressure **(*P_c > 4.5 × 10^8 atmospheres*)**

What physical processes give rise to this pressure – which are the most important ?

- **Gas pressure** P_g
- **Radiation pressure** P_r
- We shall show that P_r is negligible in stellar interiors and pressure is dominated by P_g

To do this we first need to estimate the minimum mean temperature of a star

Consider the Ω term, which is the **gravitational potential energy**:

$$-\Omega = \int_0^{M_s} \frac{GM}{r} dM \qquad (5.27)$$

Minimum mean temperature of a star

We can obtain a lower bound on the RHS by noting: at all points inside the star $r < r_s$ and hence $1/r > 1/r_s$

$$\Rightarrow \int_0^{M_s} \frac{GM}{r} dM > \int_0^{M_s} \frac{GM}{r_s} dM = \frac{GM_s^2}{2r_s} \qquad (5.28)$$

Now $dM = \rho\, dV$ and the Virial theorem can be written

$$-\Omega = 3\int_0^{V_s} P dV = 3\int_0^{M_s} \frac{P}{\rho} dM \qquad (5.28)$$

Now pressure is sum of radiation pressure and gas pressure: $P = P_g + P_r$
Assume, for now, that stars are composed of ideal gas with negligible P_r

$$P = nkT = \frac{k\rho T}{m} \qquad (5.29)$$

where n = number of particles per m^3

m = average mass of particles

k = Boltzmann's constant

The eqn of state of ideal gas

Minimum mean temperature of a star

Hence we have

$$-\Omega = 3\int_0^{M_s} \frac{P}{\rho} dM = 3\int_0^{M_s} \frac{kT}{m} dM \qquad (5.30)$$

And we may use the inequality derived above to write

$$-\Omega = 3\int_0^{M_s} \frac{kT}{m} dM > \frac{GM_s^2}{2r_s}$$

$$\Rightarrow \int_0^{M_s} TdM > \frac{GM_s^2 m}{6kr_s} \qquad (5.31)$$

We can think of the LHS as the sum of the temperatures of all the mass elements dM which make up the star
The mean temperature of the star $\bar{T}$ is then just the integral divided by the total mass of the star M_s

$$\Rightarrow M_s \bar{T} = \int_0^{M_s} TdM$$

$$\bar{T} > \frac{GM_s m}{6kr_s} \qquad (5.32)$$

Minimum mean temperature of a star

As an example for the sun we have

$$\bar{T} > 4 \times 10^6 \frac{m}{m_H} \text{ K} \qquad \text{where } m_H = 1.67 \times 10^{-27} \text{ kg} \tag{5.33}$$

Now we know that H is the most abundant element in stars and for a fully ionised hydrogen star $m/m_H = 1/2$ (as there are two particles, p + e^-, for each H atom). And for any other element m/m_H is greater

$$\bar{\mathrm{T}}_{\mathrm{Sun}} > 2 \times 10^6 \text{ K} \tag{5.33}$$

5.2.2 - Physical state of stellar material

We can also estimate the mean density of the Sun using:

$$\rho_{av} = \frac{3M_{sun}}{4\pi r_{sun}^3} = 1.4\times 10^3 \text{ kgm}^{-3} \qquad (5.34)$$

Mean density of the sun is only a little higher than water and other ordinary liquids. We know such liquids become gaseous at T much lower than $T_\odot$
Also the average K.E. of particles at $T_\odot$ is much higher than the ionisation potential of H. Thus the gas must be highly ionised, i.e. is a plasma.

It can thus withstand greater compression without deviating from an ideal gas. Note that an ideal gas demands that the distances between the particles are much greater than their sizes, and nuclear dimension is 10^{-15} m compared to atomic dimension of 10^{-10} m

Lets revisit the issue of radiation vs gas pressure. We assumed that the radiation pressure was negligible. The pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3} \qquad (5.35)$$

Where a = radiation density constant

Physical state of stellar material

Now compare gas and radiation pressure at a typical point in the Sun

$$\frac{P_r}{P_g} = \frac{aT^4}{3} \bigg/ \frac{kT\rho}{m} = \frac{maT^3}{3k\rho} \qquad (5.36)$$

Taking $T \sim T_{av} = 2\times10^6$ K, $\rho \sim \rho_{av} = 1.4\times10^3$ kgm^{-3} and $m = \dfrac{1.67\times10^{-27}}{2}$ kg

Gives $\quad \dfrac{P_r}{P_g} \sim 10^{-4}$

Hence radiation pressure appears to be negligible at a typical (average) point in the Sun.

In summary, with no knowledge of how energy is generated in stars we have been able to derive a value for the Sun's internal temperature and deduce that it is composed of a near ideal gas plasma with negligible radiation pressure.

5.2.3 - Mass dependence of radiation to gas pressure

However we shall later see that P_r does become significant in higher mass stars. To give a basic idea of this dependency: replace ρ in the ratio equation above:

$$\frac{P_r}{P_g} = \frac{maT^3}{3k\left(\frac{3M_s}{4\pi r_s^3}\right)} = \frac{4\pi ma}{9k}\frac{r_s^3 T^3}{M_s} \qquad (5.37)$$

$$\text{And from the Virial theorem}: \ \bar{T} \sim \frac{M_s}{r_s}$$

$$\Rightarrow \frac{P_r}{P_g} \propto M_s^2$$

i.e. P_r becomes more significant in higher mass stars.

5.2.4 - Energy generation in stars

So far we have only considered the dynamical properties of the star, and the state of the stellar material. We need to consider the source of the stellar energy.

Let's consider the origin of the energy i.e. the conversion of energy from some form in which it is not immediately available into some form that it can radiate.

How much energy does the sun need to generate in order to shine with it's measured flux ?

$$L_0 = 4 \times 10^{26}\ \mathrm{W} = 4 \times 10^{26}\ \mathrm{Js^{-1}}$$

Sun has not changed flux in 10^9 yr (3×10^{16} s)

$\Rightarrow$ Sun has radiated 1.2×10^{43} J

$$E = mc^2$$

$$\Rightarrow m_{lost} = 10^{26}\ \mathrm{kg} = 10^{-4} M_{sun}$$

Source of energy generation

What is the source of this energy ? Four possibilities :

- Cooling or contraction
- Chemical Reactions
- Nuclear Reactions

Cooling and contraction

These are closely related, so we consider them together. Cooling is simplest idea of all. Suppose the radiative energy of Sun is due to the Sun being much hotter when it was formed, and has since been cooling down. We can test how plausible this is.

Or is sun slowly contracting with consequent release of gravitational potential energy, which is converted to radiation ?

Source of energy generation

In an ideal gas, the thermal energy of a particle (where ***n_f = number of degrees of freedom = 3***)

$$= \frac{kT}{2} n_f$$
$$= \frac{3kT}{2} \qquad (5.38)$$

Total thermal energy per unit volume
n = number of particles per unit volume

$$= \frac{3knT}{2} \qquad (5.39)$$

$$\text{Now, Virial theorem:}$$
$$3\int_0^{V_s} P dV + \Omega = 0$$

Assume that stellar material is ideal gas (negligible P_r)

$$\Rightarrow P = nkT$$
$$3\int_0^{V_s} nkT dV + \Omega = 0 \qquad (5.40)$$

Source of energy generation

Now lets define U = integral over volume of the thermal energy per unit volume

$$\text{thermal energy per unit volume} = \frac{3knT}{2}$$

$$\Rightarrow 2U + \Omega = 0 \qquad (5.41)$$

The negative gravitational energy of a star is equal to twice its thermal energy. This means that the time for which the present thermal energy of the Sun can supply its radiation and the time for which the past release of gravitational potential energy could have supplied its present rate of radiation differ by only a factor two. We can estimate the later:

Negative gravitational potential energy of a star is related by the inequality

$$-\Omega > \frac{GM_s^2}{2r_s} \quad \text{as an approximation assume} \quad -\Omega \sim \frac{GM_s^2}{2r_s}$$

$$(5.42)$$

Source of energy generation

Total release of gravitational potential energy would have been sufficient to provide radiant energy at a rate given by the luminosity of the star L_s , for a time

$$t_{th} \sim \frac{GM_s^2}{L_s r_s} \qquad (5.43)$$

Putting in values for the Sun: $t_{\odot th} = 3 \times 10^7$ *years.*

Hence if Sun where powered by either contraction or cooling, it would have changed substantially in the last 10 million years. A factor of ~100 too short to account for the constraints on age of the Sun imposed by fossil and geological records.

Definition: t_{th} is defined as the ***thermal timescale*** *(or **Kelvin-Helmholtz timescale**)*

Chemical Reactions

Can quickly rule these out as possible energy sources for the Sun. We calculated above that we need to find a process that can produce at least 10^{-4} of the rest mass energy of the Sun. Chemical reactions such as the combustion of fossil fuels release ~ 5×10^{-10} of the rest mass energy of the fuel.

Source of energy generation

Nuclear Reactions

Hence the only known way of producing sufficiently large amounts of energy is through nuclear reactions. Two types of nuclear reactions are important for energy generation: fission and fusion. Fission reactions, such as those that occur in nuclear reactors, or atomic weapons can release ~ 5×10^{-4} of rest mass energy through fission of heavy nuclei (uranium or plutonium).

Hence we can see that both fusion and fission could in principle power the Sun.

Which is the more likely ?

As light elements are much more abundant in the solar system that heavy ones, we would expect nuclear fusion to be the dominant source.

Given the limits on $P(r)$ *and* $T(r)$ that we have just obtained - are the central conditions suitable for fusion ? We will return to this later.