6 – Stellar Structure II

6.1 - Non-relativistic Fermi gas

The number of electron spin states are $g = 2s + 1 = 2$, since for electrons $s = 2$ (spin-up and spin-down) .

According to **de Broglie**, the electron wavelength decreases with increasing momentum, and thus with the kinetic energy.

Since the kinetic energy is proportional to the temperature, T, for small enough T, wave functions overlap, and **quantum statistics** becomes important.

Let us consider $T = 0$. This is a situation in which the electrons should be standing still. But we know from the uncertainty principle that this is never possible. There will be always a "zero" motion for the electrons.

According to the Pauli principle, one cannot accommodate more than 1 (2 for $g = 2$) electrons in each state. The first electron goes to lowest state, the second must occupy a higher energy state, and so on, until all the electrons are allocated in energy states.

Thus, at $T = 0$, all the states up to energy $E = E_F$ are filled, and at $E > E_F$ they are empty, so that the distribution of electrons in energy states is given by

$$f(E) = \begin{cases} 1, & E \le E_F \\ 0, & E > E_F \end{cases} \qquad (6.1)$$

6.1.2 - Density of states

In order to proceed we need to know how many states exist within an energy interval between E and E + dE (**density of states**).

Let us assume that the region to which the electrons are limited to is the interior of a cube. The final results will be independent of this hypothesis. In this way, the electron wqavefunction Ψ will have to satisfy the boundary conditions Ψ(x, y, z) = 0 for x = 0, y = 0, z = 0 and x = a, y = a, z = a, where a is the side of the cube. The solution is given by

$$\Psi(x,y,z) = A\sin(k_x x)\sin(k_y y)\sin(k_z z) \qquad (6.2)$$

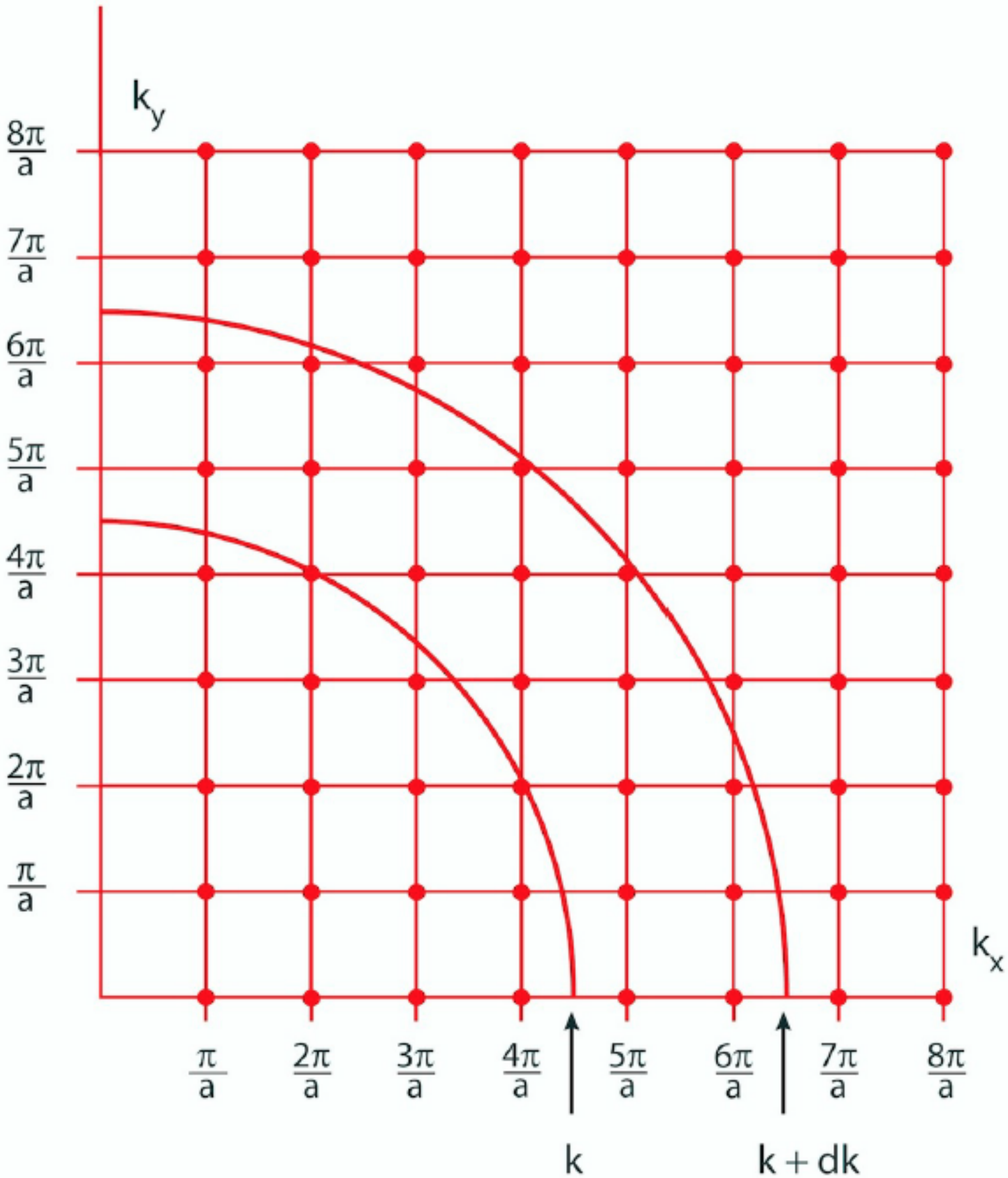
with

$$k_x a = n_x \pi, \quad k_y a = n_y \pi, \quad k_z a = n_z \pi \qquad (6.3)$$

where n_x, n_y and n_z are positive integers and *A* is a normalization constant.
For each group (n_x, n_y, n_z) we have an energy

$$E\left(n_x, n_y, n_z\right) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right) = \frac{\hbar^2 \pi^2}{2ma^2} n^2 \qquad (6.4)$$

Density of states



Allowed states in part of momentum space contained in the plane $\mathbf{k_x}$ - $\mathbf{k_y}$.

Each state is represented by a point in the lattice.

Density of states

where $$n^2 = n_x^2 + n_y^2 + n_z^2 \qquad (6.5)$$

(not to be confused with particle density!).

Equations (6.2 – 6.4) represent the quantization of a particle in a box, where k ≡ (k_x, k_y, k_z) is the momentum (divided by ħ) of the particle in the box. Due to the Pauli principle, a *given* momentum can only be occupied by at most two electrons with opposite spins.

Consider the space of vectors **k**: for each cube of side length π/a there exists, in this space, only one point that represents a possible solution in the form (6.2). The possible number of solutions (see figure in previous slide) n(k) with the magnitude of k between k and k + dk is given by the ratio between the volume of the region shown in the figure and the volume $(\pi/a)^3$ for each allowed solution in the k-space. One obtains,

$$dn(k) = \frac{1}{8} 4\pi k^2 dk \frac{1}{(\pi / a)^3} \qquad (6.6)$$

Density of states

where $4\pi k^2 dk$ is the volume of a spherical shell in the k-space with radius between k and k + dk. Only 1/8 of the shell is considered, since only positive values of k_x, k_y and k_z are necessary for counting all the states with eigenfunctions defined by Eq. (6.2). With the aid of Eq. (6.4) we can make the energy appear explicitly in Eq. (6.6):

$$dn(E) = \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2} dE \qquad (6.7)$$

There is a factor $V = a^3$ in this and the following formulas that I will hide for convenience, bring it back to life later.

The total number of possible states of the system is obtained by integrating the product f(E) dn/dE from 0 to the minimum value of the energy needed to include all the N electrons. This value, E_F , is called the **Fermi energy**. Thus, we obtain

$$N = \int_0^\infty \frac{dn}{dE}(E) f(E)\, dE = \int_0^{E_F} \frac{dn}{dE}(E)\, dE$$
$$= \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^{E_F} \sqrt{E}\, dE = \frac{1}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left(E_F\right)^{3/2} \qquad (6.8)$$

6.1.3 - Fermi Energy

Inverting Eq. (6.8) we obtain

$$E_F = \frac{\hbar^2}{2m}\left(3\pi^2 n_e\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3}{\pi}n_e\right)^{2/3} \qquad (6.9)$$

where $n_e = N/a^3$ is the electron number density.

This value, E_F , is called the **Fermi energy**.

For example, in solids electrons in the conduction band might be assumed free and described by wavefuncitons of the kind (6.2). Then we can use eq. (6.9) to obtain (using $n_e = 10^{29}$ electrons/m^3).

$$E_F = \frac{h^2}{8m}\left(\frac{3}{\pi}n\right)^{2/3} \approx \frac{\left(6.6\cdot 10^{-34}\right)^2}{8\cdot 9\cdot 10^{-31}}\left(1\cdot 10^{29}\right)^{2/3}\ \mathrm{J} \approx 1\cdot 10^{-18}\ \mathrm{J} \approx 6\ \mathrm{eV} \qquad (6.10)$$

and a "**Fermi temperature**" given by

$$T_F \equiv E_F / k_B \approx \text{ few eV} \approx \text{ few } 10^4\ \mathrm{K} \qquad (6.11)$$

Thus, at room temperature, this Fermi gas is strongly **degenerate** ($E_F >> k_B T$).

Fermi energy

Note that the Fermi energy only depends on the number density of the confined electrons. The mean energy of the electrons is given by

$$\bar{E} = \frac{2}{N}\int_0^{\varepsilon_F} E\frac{dn}{dE}(E)\,dE = \frac{3}{5}E_F \qquad (6.12)$$

Now, according to classical physics, the mean thermal energy of the electrons is 3kT/2, where T is the electron gas temperature. Thus, if $k_BT \ll E_F$ then our original assumption that the electrons are cold is valid. Note that, in this case, the electron energy is much larger than that predicted by classical physics. Electrons in this state are termed **degenerate**. On the other hand, if $k_BT \gg E_F$ then the electrons are hot, and are essentially governed by classical physics. Electrons in this state are termed **non-degenerate**.

The total energy of a degenerate electron gas is

$$E = N\bar{E} = \frac{3}{5}NE_F \qquad (6.13)$$

6.1.4 - Pressure of a Fermi gas

The internal energy and pressure of an ideal gas goes to 0 as $T \to 0$. This is not the case for a degenerate Fermi gas !

$$E = \frac{3}{5} N E_F \qquad (6.14)$$

On the other hand, $E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$ and

$$E = \frac{3}{10}\frac{\hbar^2 N}{m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} = A\, N^{5/3} V^{-2/3}$$

At T = 0, thermodynamics tell us that there is no distinction between the free energy F and the internal energy E: Thus,

$$P = -\left(\frac{\partial E}{\partial V}\right)_N = \frac{2}{3} V^{-5/3} \frac{3}{10}\frac{\hbar^2 N}{m}\left(3\pi^2 \frac{N}{V}\right)^{2/3} = \frac{1}{5}\frac{\hbar^2 n_e}{m}\left(3\pi^2 n_e\right)^{2/3} = \frac{2}{5} n_e E_F \qquad (6.15)$$

$$P = \frac{1}{20}\frac{h^2}{m}\left(\frac{3}{\pi}\right)^{2/3} n_e^{5/3} \quad \text{or} \quad PV = \frac{2}{3}E \qquad (6.16)$$

It looks just as for *an ideal gas*. This is a direct consequence of the quadratic relation between energy and momentum.

However, this is a non-zero pressure at T = **0**, which does not depend on T. Let's estimate this pressure for a typical metal:

$$P = \frac{2}{5} n E_F \approx 10^{29}\ \text{m}^{-3} \times 5 \times 10^{-19}\ \text{J} = 5 \times 10^{10}\ \text{Pa}$$

In metals, this enormous pressure is counteracted by the Coulomb attraction of the electrons to the positive ions.

6.2 - Relativistic Fermi gas

In this case, let us first rewrite Eq. (6.6) as

$$dn = \frac{d^3k d^3r}{(2\pi)^3} = \frac{d^3p d^3r}{(2\pi\hbar)^3} \qquad (6.17)$$

where we **used** $4\pi k^2 dk = d\Omega_k k^2 dk = d^3k$ and $a^3 = d^3r$.
The above equation is very important. It tells us that the number of states available within a cell in the phase space (**p**, **r**) with "volume" d^3pd^3r is given by d^3pd^3r/h. That is, the Planck constant h is the measure of how many states are within d^3pd^3r. Think about it as a three dimensional representation of Heisenberg principle $dxdp/h \sim 1$. Eq. (6.17) is valid for Fermions (electrons, nucleons, etc.) and for Bosons (photons, pions, etc.)

For a relativistic electron, its momentum is much larger than its mass. Thus,

$$E = \sqrt{p^2c^2 + m^2c^4} \sim pc \qquad (6.18)$$

It is now left as an exercise to show that Eq. (6.8) and the following yield $E \sim \rho^{4/3}$ and $P \sim \rho^{4/3}$.

In summary:

$$\begin{aligned} P &\sim \rho^{5/3} \quad \text{for non-relativistic electrons} \\ &\sim \rho^{4/3} \quad \text{for relativistic electrons} \end{aligned} \qquad (6.19)$$

6.3 - Equation of state $P = P(\rho)$ and polytropic models

Polytropic equation of state (EOS):

$$P = \kappa \rho^{\Gamma} \qquad (6.20)$$

As we saw in previous slides, **non-relativistic electron gas** ($\rho < 10^6$ g/cm^3)

Γ = 5/3;

$$\kappa = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{\hbar^2 \left(N_A Y_e\right)^{5/3}}{m_e} \qquad (6.21)$$

Where N_A is the Avogadro number and Y_e is the fraction number of electrons.

For a **relativistic electron gas** ($\rho > 10^6$ g/cm^3) this relation will change to

Γ = 4/3;

$$\kappa = \left(\frac{3}{\pi}\right)^{1/3} \frac{\hbar c \left(N_A Y_e\right)^{4/3}}{8} \qquad (6.22)$$

We will now derive the **Lane-Emden equations**:

With help of the hydrostatic equilibrium and mass conservation equations we get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G \rho \qquad (6.23)$$

6.3.1 - Lane-Emden Eaqutions

Using $P = \kappa \rho^{\Gamma}$ with $\Gamma = 1 + \frac{1}{n}$, we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n \qquad (6.24)$$

where $\rho = \rho_c \theta^n = \rho(r=0)\theta^n$ and $\xi = ar$ with

$$a = \left[\frac{(n+1)\kappa\rho_c^{\frac{1}{n}-1}}{4\pi G}\right]^{-1/2}$$

The boundary conditions are $\theta(0) = 1,\qquad \theta'(0) = 0$

(6.25)

Analytical solution of Lane-Emden equations:

n = 0 → $\theta(\xi) = 1 - \xi^2/6$

n = 1 → $\theta(\xi) = \frac{\sin\xi}{\xi}$ (6.26)

n = 5 → $\theta(\xi) = \left(1 - \xi^2/3\right)^{-1/2}$

The solutions have a maximum at ξ, r = 0.

For n < 5, $\xi_n = aR$ is finite. For n > 5, ξ_n is infinite.

6.3.2 - Central density in polytropic models

$$m(r) = \int_0^r 4\pi\rho\, r'^2\, dr' = 4\pi\rho_c \frac{r^3}{\xi^3}\int_0^{\xi} \theta^n \xi'^2 d\xi' = 4\pi\rho_c r^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right) \qquad (6.27)$$

Since $\frac{r}{\xi} = a^{-1} = \frac{R}{\xi_n}$ we get

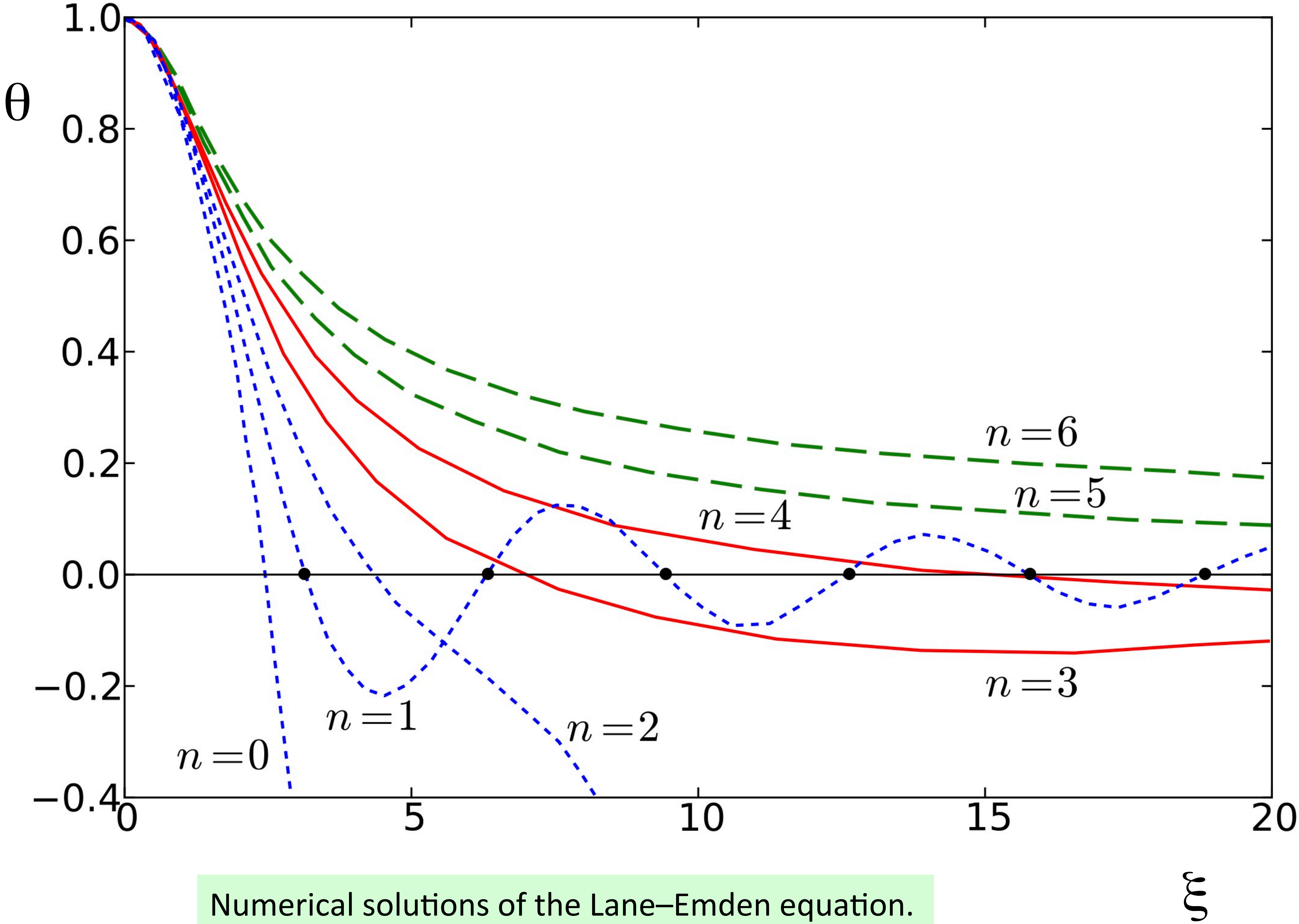
$$\mathrm{M} = 4\pi\rho_c \mathrm{R}^3\left(-\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_n} \qquad (6.28)$$

We now define the average density $\bar{\rho} = \frac{3M}{4\pi R^3}$ and get

$$\frac{\bar{\rho}}{\rho_c} = \left(-\frac{3}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)_{\xi=\xi_n} \qquad (6.29)$$

n	ξ_n	$\rho_c/\langle\rho\rangle$
0	2.4494	1.000
1	3.14159	3.28987
2	4.35287	11.40254
3	6.89685	54.1825
4	14.97155	622.408
5	∞	∞

Polytropic models



Numerical solutions of the Lane–Emden equation.

6.3.4 - Constructing a polytropic model

1 – Solve Lane-Emden equations

2 – Use M, R to obtain $\bar{\rho},\ \rho_c$

3 – Use $a^{-1} = r/\xi = R/\xi_n$ to adjust ξ scale

4 – Knowing a and ρ_c allows to determine κ
Then

$$P(r) = \kappa\rho^{(n+1)/n} = \kappa\rho_c^{(n+1)/n}\theta^{n+1} \qquad (6.30)$$

5 – Get

$$m(r) = 4\pi\rho_c r^3\left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right) \qquad (6.31)$$

6 - κ is a free parameter. If κ is fixed, one can only construct models for given M or R, and fixed n.

Ex: the Sun

$$M_\Theta = 1.989 \times 10^{33}\ g; \qquad R_\Theta = 6.96 \times 10^{10}\ cm \qquad (6.32)$$

- From table $\xi_{n=3}$ = 6.897; $\frac{\rho_c}{<\rho>} = 54.18$ → ρ_c = 76.39 g/cm³, $\langle\rho\rangle$ = 1.41 g/cm³

and $a^{-1} = R/\xi_3 = 1.01 \times 10^{10}$

- From this value of a, we get $\kappa = 3.85 \times 10^{14}$ → $P_c = \kappa \rho_c^{\Gamma} = 1.24 \times 10^{17}\ dyn/cm^2$

(6.33)

(6.34)

- Using ideal gas law $P = nkT$ → $T_c = 1.2 \times 10^7$ K (6.35)

-A detailed calculation (e.g. **The Standard Solar Model** of John Bahcall) yields

$T_c = 1.57 \times 10^7$ K and → ρ_c = 156 g/cm³ (6.36)

6.3.5 - Polytropic model with fixed n and κ (and ρ_c)

For a polytropic model we have

$$\left(\frac{r}{\xi}\right)^2 = a^{-2} = \frac{(n+1)\kappa\rho_c^{\frac{1}{n}-1}}{4\pi G} = \left(\frac{R}{\xi_n}\right)^2 \qquad (6.37)$$

→ $R \sim \rho_c^{\frac{1-n}{2n}}$

As long as n > 1, the radius becomes smaller with increasing ρ_c.

→ $$M = 4\pi\rho_c R^3 \left(-\frac{1}{\xi}\frac{d\theta}{d\xi}\right)_{\xi=\xi_n} = C_1 \rho_c^{(3-n)/2n}$$

$$C_1 = 4\pi\left(-\frac{\theta'}{\xi}\right)_{\xi=\xi_n} \xi_n^{\;3} \left(\frac{n+1}{4\pi G}\right)^{3/2} \kappa^{3/2}$$

→ $$R \sim M^{(1-n)/(3-n)} \qquad (6.38)$$

which is useful for many estimates.

6.3.6 - Application: Chandrasehkhar Mass

EOS of ultrarelativistic electron gas, $P \sim \rho_c^{4/3}$, is a polytrope with n=3

→

$$M = C_1 \rho_c^{(3-n)/2n} = C_1 = 4\pi \left(-\frac{\theta'}{\xi} \right)_{\xi_3} \xi_3^{\,3} \left(\frac{\kappa}{\pi G} \right)^{3/2} = const. \qquad (6.38)$$

This is called the **Chandrasekhar mass**:

$$M_{ch} = 1.457 \left(2Y_e \right) M_\Theta \qquad (6.39)$$

It is the maximum mass for which a star with an ultrarelativistic electron EOS is stable.

It applies to **White Dwarfs**:

- White dwarfs are composed of carbon, oxygen and neon nuclei.

- They have the radius of 1/100 solar radius

- They are the fatal fate of the main-sequence stars, with $M < 8\ M_\Theta$

- Mass is dominated by nuclei, while pressure is given by electrons.

6.3.7 - Chandrasehkhar Mass (back of the envelope)

Suppose N fermions in star with radius R:

→ Number density **n = N/V ~ N/R³**

Heisenberg principle: $p \sim \hbar n^{1/3}$ Fermi energy: $E_F \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}$

Gravitational energy per fermion: $E_g \sim -G\frac{M m_B}{R}$; $m_B = \frac{M}{N}$

Equilibrium: $$\mathrm{E} = \mathrm{E(R)} = \frac{\hbar \mathrm{c} \mathrm{N}^{1/3}}{\mathrm{R}} - \frac{\mathrm{G N m_B^2}}{\mathrm{R}} = \text{minimum}$$

→ minimum occurs at $N = N_0$ and at finite $R = R_0$

→ $$\frac{\hbar c N_0^{\ 1/3}}{R_0} = \frac{G N_0 m_B^2}{R_0}$$ → $$N_0 \sim \left(\frac{\hbar c}{G m_B}\right)^{3/2} \sim 2\times 10^{57}$$

→ $$M_0 \sim N_0 m_B \sim 1.5 M_\Theta \qquad (6.40)$$

Chandrasehkhar Mass (back of the envelope)

Assume equilibrium is related to the onset of relativistic degeneracy of particles of mass m ($E_F > mc^2$):

$$E_F \sim \frac{\hbar c N_0^{1/3}}{R_0} \quad \Rightarrow \quad R_0 < \frac{\hbar}{mc}\left(\frac{\hbar c}{G m_B^2}\right)^{1/2}$$

While pressure can come from electrons ($m = m_e$) or neutrons ($m = m_B$), the mass is always given by baryons (m_B). Then

$R_0 \sim 5\times 10^8$ cm for ($m = m_e$) or $R_0 \sim 3\times 10^5$ cm for ($m = m_B$).

These are typical radii for **white dwarfs** (degenerate electrons) or **neutron stars** (degenerate nucleons).

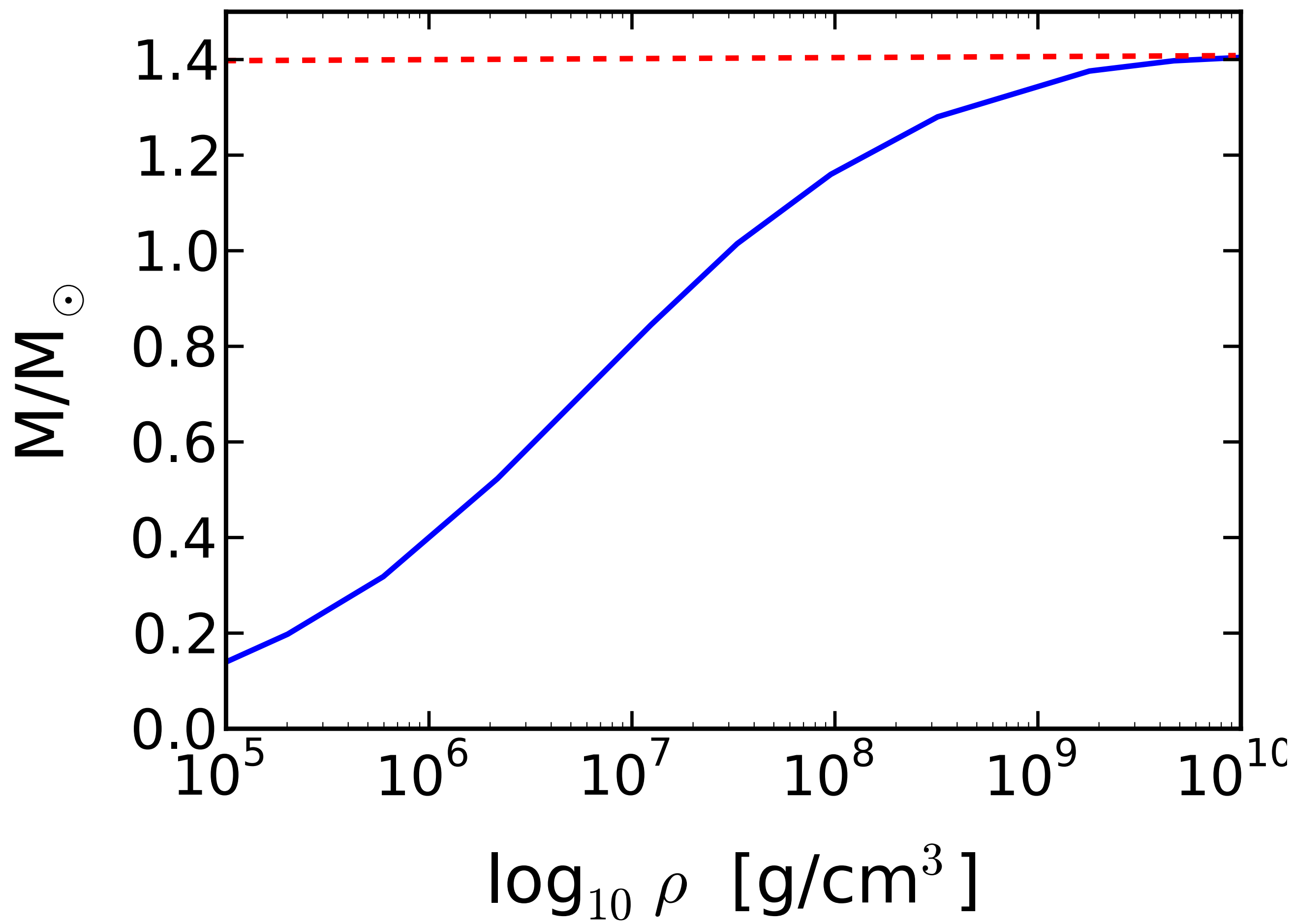
Polytrope with Γ = 5/3

For non-relativistic electrons, with $\Gamma = 5/3$, we can solve Eqs. (5.6) and (5.8) together with

$$\frac{dP}{dr} = \frac{d\rho}{dr}\frac{dP}{d\rho}$$

with $dP/d\rho$ obtained from the non-relativistic EOS, Eq. (6.15).

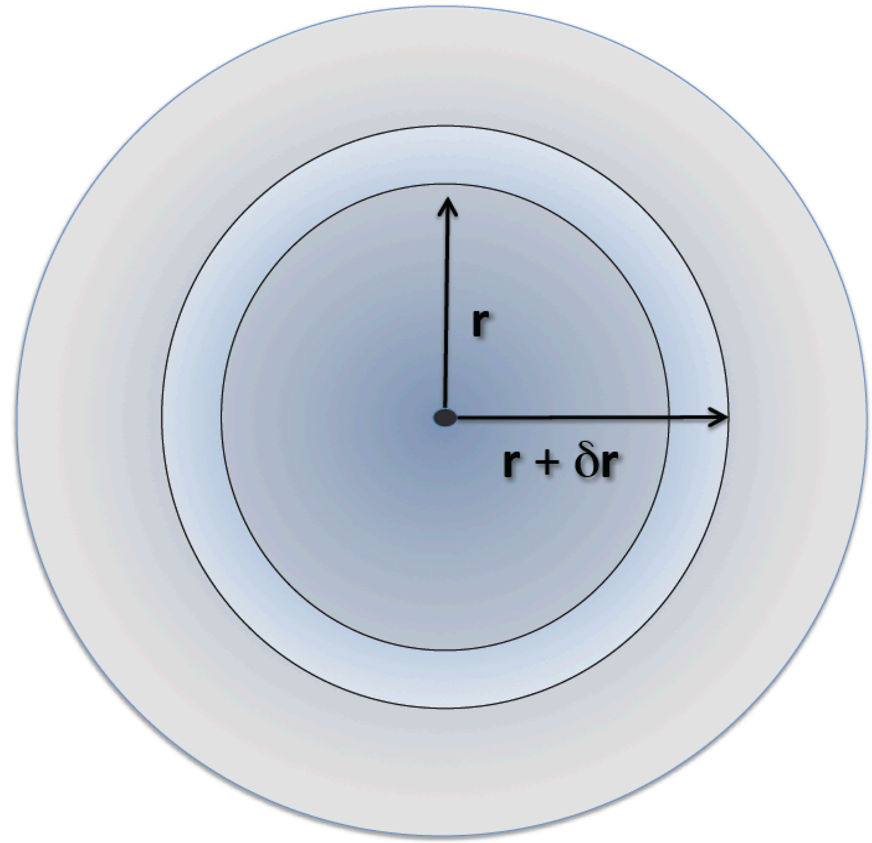
The solution obtained numerically as a function of the central density and is given below. It also yields a maximum mass of ~ 1.4 $M_\odot$. The agreement with our simplified discussions is remarkable.



6.4 - Equation of energy production

The third equation of stellar structure: relation between energy release and the rate of energy transport

Consider a spherically symmetric star in which energy transport is radial and in which time variations are unimportant.



L(r) = rate of energy flow across sphere of radius r

L(r + δ r) = rate of energy flow across sphere of radius r + δ r

Because shell is thin:

$$\delta V(r) = 4\pi r^2 \delta r$$

$$\text{and} \quad \delta m(r) = 4\pi r^2 \rho(r) \delta r \qquad (6.41)$$

Energy production

We define ε = ***energy release per unit mass per unit volume*** **(W kg^{-1})**
Hence energy release in shell is written:

$$4\pi r^2 \rho(r)\delta r \varepsilon$$

Conservation of energy leads us to

$$L(r+\delta r) = L(r) + 4\pi r^2 \rho(r)\delta r \varepsilon$$

$$\Rightarrow$$

$$\frac{L(r+\delta r) - L(r)}{\delta r} = 4\pi r^2 \rho(r)\delta r \varepsilon$$

and for $\delta r \to 0$,

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon \qquad (6.42)$$

This is the ***equation of energy production.***

We now have three of the equations of stellar structure. However we have five unknowns *P(r), M(r), L(r),* $\rho(r)$, $\varepsilon(r)$. In order to make further progress we need to consider energy transport in stars.

6.5 - Energy transport

There are three ways energy can be transported in stars

- **Convection** – energy transport by mass motions of the gas
- **Conduction** – by exchange of energy during collisions of gas particles (usually e^-)
- **Radiation** – energy transport by the emission and absorption of photons

Conduction and radiation are similar processes – they both involve transfer of energy by direct interaction, either between particles or between photons and particles.

Which is the more dominant in stars ?
Energy carried by a typical particle ~ $3kT/2$ is comparable to energy carried by typical photon ~ hc/λ

But **number density of particles is much greater than that of photons**. This would imply conduction is more important than radiation. **BUT**,

Mean free path of photon ~ 10^{-2} m

Mean free path of particle ~ 10^{-10} m

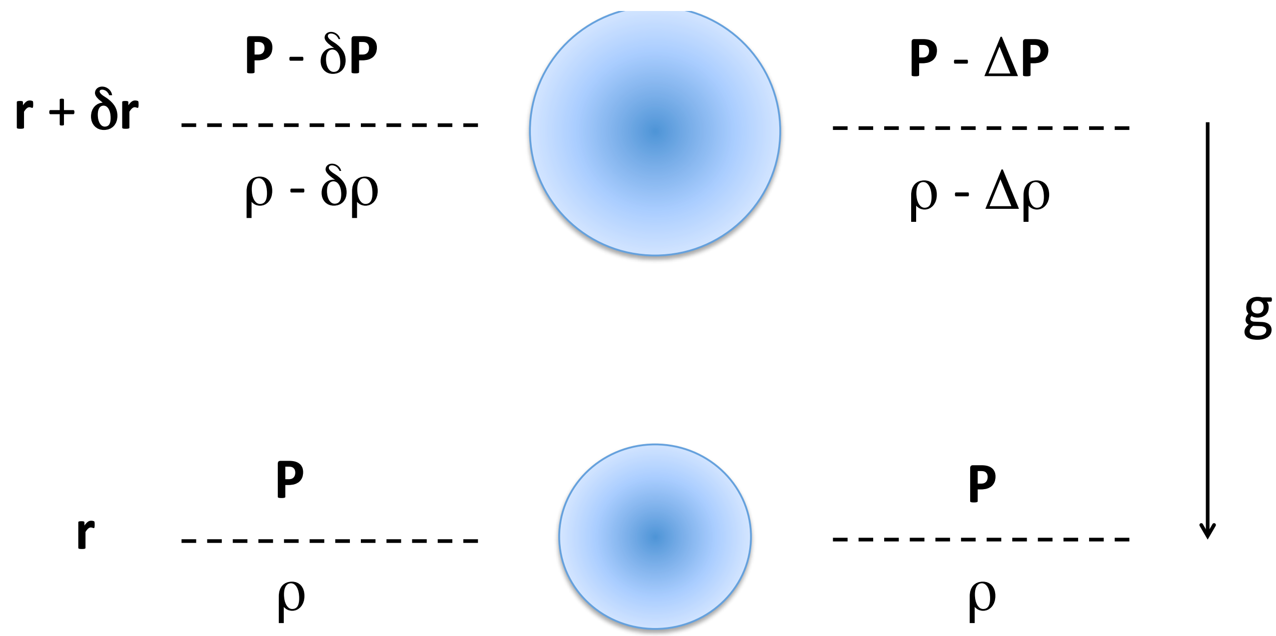
Photons can move across temperature gradients more easily, hence larger transport of energy. **Conduction is negligible, radiation transport in dominant.**

6.5.1 - Convection

Convective element of stellar material

Convection is the mass motion of gas elements – only occurs when temperature gradient exceeds some critical value. We can derive an expression for this.

Consider a convective element at distance *r* from center of star. Element is in equilibrium with surroundings



Now let's suppose it rises to $r + \delta r$. It expands, $P(r)$ and $\rho(r)$ are reduced to $P - \delta P$ and $\rho - \delta\rho$

But these may not be the same as the same as the new surrounding gas conditions. Define those as $P - \Delta P$ and $\rho - \Delta\rho$

If gas element is denser than surroundings at $r + \delta r$ then will sink (i.e. stable)
If it is less dense then it will keep on rising – *convectively unstable*

Convection

The condition for instability is therefore

$$\rho - \delta\rho < \rho - \Delta\rho \qquad (6.43)$$

Whether or not this condition is satisfied depends on two things:

- The rate at which the element expands due to decreasing pressure
- The rate at which the density of the surroundings decreases with height

Let's make two assumptions

1. The element rises *adiabatically*
2. The element rises at a speed much less than the sound speed. During motion, sound waves have time to smooth out the pressure differences between the element and the surroundings. Hence $\delta P = \Delta P$ *at all times*

The first assumption means that the element must obey the **adiabatic relation between pressure and volume**

$$PV^{\gamma} = \text{constant} \qquad (6.44)$$

Where $\gamma = c_p/c_v$ is the **specific heat** (i.e. the energy in J to raise temperature of 1kg of material by 1K) at constant pressure, divided by specific heat at constant volume.

Convection

Given that *V* is inversely proportional to ρ , we can write

$$\frac{P}{\rho^{\gamma}} = \text{constant} \qquad (6.45)$$

Hence equating the term at *r and r + δ r*:

$$\frac{P - \delta P}{(\rho - \delta\rho)^{\gamma}} = \frac{P}{\rho^{\gamma}} \qquad (6.46)$$

If δρ is small we can expand $(\rho - \delta\rho)^{\gamma}$ using the binomial theorem as follows

$$(\rho - \delta\rho)^{\gamma} \sim \rho^{\gamma} - \gamma\rho^{\gamma-1}\,\delta\rho$$ and combining last two expressions

$$\delta\rho = \frac{\rho}{\gamma P}\delta P$$

Now we need to evaluate the change in density of the surroundings, $\Delta\rho$
Let's consider an infinitesimal rise of δr

$$\Delta\rho = \frac{d\rho}{dr}\delta r \qquad (6.47)$$

Convection

and substituting these expressions for $\delta\rho$ and $\Delta\rho$ into the condition for convective instability derived above:

$$\frac{\rho}{\gamma P}\delta P < \frac{d\rho}{dr}\delta r \qquad (6.48)$$

This can be rewritten by recalling our 2nd assumption that element will remain at the same pressure as it surroundings, so that in the limit

$$\delta r \to 0, \quad \frac{\delta P}{\delta r} = \frac{dP}{dr} \qquad (6.49)$$

$$\frac{\rho}{\gamma P}\frac{dP}{dr} < \frac{d\rho}{dr} \qquad (6.50)$$

The LHS is the density gradient that would exist in the surroundings if they had an adiabatic relation between density and pressure. RHS is the actual density in the surroundings. We can convert this to a more useful expression, by first dividing both sides by dP/dr. Note that dP/dr is negative, hence the inequality sign must change.

Convection

$$\frac{\rho}{\gamma P} < \frac{d\rho}{dr} \bigg/ \frac{dP}{dr}$$

$$\Rightarrow \frac{\rho}{\gamma P} > \frac{d\rho}{dP}$$

or

$$\left(\frac{P}{\rho}\right)\frac{d\rho}{dP} < \frac{1}{\gamma} \qquad (6.51)$$

And for an ideal gas in which radiation pressure is negligible (where m is the mean mass of particles in the stellar material)

$$P = \frac{\rho k T}{m} \qquad (6.52)$$

$$\ln P = \ln \rho + \ln T + \text{constant}$$

And can differentiate to give

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \qquad (6.53)$$

And combining this with the equation above gives ....

Condition for occurrence of convection

$$\frac{P}{T}\frac{dT}{dP} > \frac{\gamma - 1}{\gamma} \qquad (6.54)$$

Which is the condition for the occurrence of convection (in terms of the temperature gradient). A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient.

If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

The criterion can be satisfied in two ways. The ratio of specific heats γ is close to unity or the temperature gradient is very steep.

For example if a large amount of energy is released at the center of a star, it may require a large temperature gradient to carry the energy away. Hence where nuclear energy is being released, convection may occur.

Condition for occurrence of convection

Alternatively in the cool outer layers of a star, *gas* may only be partially ionised, hence much of the heat used to raise the temperature of the *gas goes* into ionization and hence the specific heat of the *gas* at constant V is nearly the same as the specific heat at constant P, and $\gamma \sim 1$.

In such a case, a star can have a cool outer convective layer.

Convection is an extremely complicated subject and it is true to say that the **lack of a good theory of convection** is one of the worst defects in our present studies of stellar structure and evolution. We know the conditions under which convection is likely to occur but don't know how much energy is carried by convection.

Fortunately we will see that we can often find occasions where we can manage without this knowledge.

6.6 - The characteristic timescales

There are 3 characteristic timescales that aid concepts in stellar evolution

The dynamical timescale

For the Sun $t_d \sim 2000$ s

$$t_d = \left(\frac{2r^3}{GM}\right)^{\frac{1}{2}} \qquad (6.55)$$

The thermal timescale

Time for a star to emit its entire reserve of thermal energy upon contraction provided it maintains constant luminosity (Kelvin-Helmholtz timescale)

For the Sun $t_{th} \sim 30$ Myrs

$$t_{th} \sim \frac{GM^2}{Lr} \qquad (6.56)$$

The nuclear timescale

Time for star to consume all its available nuclear energy (ε = typical nucleon binding energy/nucleon rest mass energy

For Sun t_{nuc} is larger than age of Universe

$$t_{nuc} \sim \frac{\varepsilon M c^2}{L} \qquad (6.57)$$

$$\Rightarrow t_d << \quad t_{th} \quad << t_{nuc} \qquad (6.58)$$