

# **7 - *Stellar Structure III***

# Fundamental physical constants

$a$	radiation density constant	$7.55 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$
$c$	velocity of light	$3.00 \times 10^8 \text{ m s}^{-1}$
$G$	gravitational constant	$6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$
$h$	Planck's constant	$6.62 \times 10^{-34} \text{ J s}$
$K$	Boltzmann's constant	$1.38 \times 10^{-23} \text{ J K}^{-1}$
$m_e$	mass of electron	$9.11 \times 10^{-31} \text{ kg}$
$m_H$	mass of hydrogen atom	$1.67 \times 10^{-27} \text{ kg}$
$N_A$	Avogadro's number	$6.02 \times 10^{23} \text{ mol}^{-1}$
$\sigma$	Stefan Boltzmann constant	$5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$ ( $\sigma = ac/4$ )
$R$	gas constant ( $k/mH$ )	$8.26 \times 10^3 \text{ J K}^{-1} \text{ kg}^{-1}$
$e$	charge of electron	$1.60 \times 10^{-19} \text{ C}$
$L_{\odot}$	luminosity of Sun	$3.86 \times 10^{26} \text{ W}$
$M_{\odot}$	mass of Sun	$1.99 \times 10^{30} \text{ kg}$
$T_{eff\odot}$	effective temperature of sun	5780 K
$R_{\odot}$	radius of Sun	$6.96 \times 10^8 \text{ m}$
<i>Parsec</i>	<i>(unit of distance)</i>	$3.09 \times 10^{16} \text{ m}$

## 7.1 - The equation of radiative transport

We assume for the moment that the condition for convection is not satisfied, and we will derive an expression relating the change in temperature with radius in a star assuming all energy is transported by radiation. Hence we ignore the effects of convection and conduction.

The equation of radiative transport with gas conditions is a function of only one coordinate, in this case  $r$ .

Photons interact with free electrons, cause atomic transitions and ionize atoms.

The net radiation flux is given by:

$$F_{rad} = -\frac{4\pi}{3} \frac{1}{\kappa\rho} \frac{dI}{dr} = -\frac{4\pi}{3} \frac{1}{\kappa\rho} \frac{d(\sigma T^4)}{dr} \quad (6.59)$$

Where  $I$  is the intensity of radiation and  $\kappa$  is the opacity.

# Rosseland mean opacity

Assume  $\beta = \frac{P_{gas}}{P}$  where  $P = P_{gas} + P_{rad}$  for a star composed of gas-

$$(7.1)$$

$$(7.2)$$

radiation mixture.

Using the Stefan-Boltzmann law,  $L = 4\pi r^2 \sigma T^4$ , one can show that the radiation pressure due to a photon gas is

$$P_{rad} = \frac{aT^4}{3} \quad (7.3) \quad \text{where} \quad a = \frac{4\sigma}{3c} \quad (7.4) \quad \text{Thus,} \quad 1 - \beta = \frac{P_{rad}}{P} = \frac{aT^4}{3P} \quad (7.5)$$

Then

$$L = -4\pi r^2 \left( \frac{ac}{3\rho\kappa} \right) \frac{dT^4}{dr} = -4\pi r^2 \left( \frac{c}{\rho\kappa} \right) \frac{dP_{rad}}{dr} \quad (7.6)$$

Now we average over all radiation frequencies  $\nu$ :

$$\bar{F} = \left\langle \frac{L}{4\pi r^2} \right\rangle_\nu = -\frac{4\pi}{3} \frac{1}{\bar{\kappa}\rho} \frac{dI}{dr}$$

with  $\bar{\kappa}$  being the **Rosseland mean opacity**

$$(7.7)$$

$$\frac{1}{\bar{\kappa}} = \frac{\pi}{acT^3} \int_0^\infty \frac{1}{\kappa} \frac{dI(\nu, T)}{dT} d\nu \quad (7.8)$$

# Radiation transport

Where  $I(\nu, T)$  is the Planck function for the intensity of blackbody radiation:

$$I(\nu, T) = \frac{2h}{c^2} \frac{\nu^3}{e^{h\nu/kT} - 1} \quad (7.9)$$

We thus can write

$$F_{rad} = \frac{L(r)}{4\pi r^2} = -\frac{4}{3} \frac{ac}{\bar{\kappa}\rho} T^3 \frac{dT}{dr} \quad (7.10)$$

or

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\bar{\kappa}\rho}{T^3} \frac{L(r)}{4\pi r^2} \quad (7.11)$$

and this is the **equation for radiation transport**.

# 7.2 - From atoms to atmospheres

## Chemical composition

How can we describe the relative abundances of elements (nuclei) of different species and their evolution in a given sample (say, a star, or the Universe)?

### Number density

We could use the number density  $n_i$  = number of nuclei of species  $i$  per  $\text{cm}^3$

**Disadvantage:** tracks not only nuclear processes that create or destroy nuclei, but also density changes, for example due to compression or expansion of the material.

### Mass fraction

$X_i$  is fraction of total mass of sample that is made up by nucleus of species  $i$

$$(7.12) \quad n_i = \frac{X_i \rho}{m_i}$$

$\rho$  : mass density ( $\text{g}/\text{cm}^3$ )

$m_i$  mass of nucleus of species  $i$

(CGS only !!!)

with  $m_i \approx A_i \cdot m_u$  and

(7.13)

$$m_u = m_{12C} / 12 = 1 / N_A \quad \text{as atomic mass unit (AMU)}$$

(7.14)

# Mass fraction and abundance

$$n_i = \left( \frac{X_i}{A_i} \right) \rho N_A \quad (7.15)$$

call this abundance  $Y_i$

**note:** if the mass is expressed in the CGS unit (grams) then  $1 \text{ [g]} = N_A \cdot m_u$

(we neglect the binding energies and the mass of the electrons in the atoms)

so 
$$n_i = Y_i \rho N_A \quad (7.16)$$

with

$$Y_i = \frac{X_i}{A_i} \quad (7.17)$$

**note:** Abundance has no units only valid in CGS

The abundance  $Y$  is proportional to number density but changes only if the nuclear species gets destroyed or produced. Changes in density are factored out.

of course

$$\sum_i X_i = 1 \quad (7.18)$$

but, since  $Y_i = X_i/A_i < X_i$

$$\sum_i Y_i < 1 \quad (7.19)$$

# Mass fraction and abundance

Mean molecular weight  $\mu_i$

= average mass number =

$$(7.21) \quad \mu_i = \frac{1}{\sum_i Y_i}$$

$$\langle A \rangle = \frac{\sum_i A_i n_i}{\sum_i n_i} = \frac{\sum_i A_i Y_i}{\sum_i Y_i} = \frac{1}{\sum_i Y_i} \quad \text{or}$$

(7.20)

For 100% hydrogen:  $Y_e = 1$   
 For equal number of protons and neutrons ( $N = Z$  nuclei):  $Y_e = 0.5$   
 For pure neutron gas:  $Y_e = 0$

## Electron Abundance $Y_e$

As matter is electrically neutral, for each nucleus with charge number  $Z$  there are  $Z$  electrons:

$$Y_e = \sum_i Z_i Y_i$$

(7.21)

can also write:

$$Y_e = \frac{\sum_i Z_i Y_i}{\sum_i A_i Y_i}$$

(7.23)

and as with nuclei, electron density

$$n_e = \rho N_A Y_e$$

(7.22)

prop. to number of protons

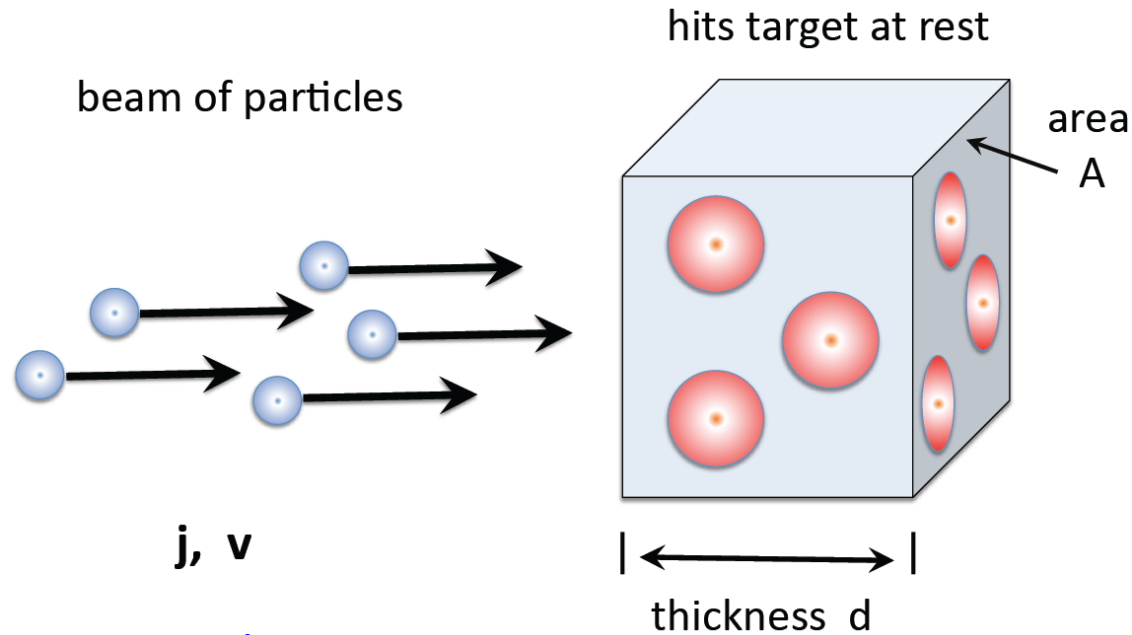
prop. to number of nucleons

So  $Y_e$  is ratio of protons to nucleons in sample (counting all protons including the ones contained in nuclei - not just free protons as described by the “proton abundance”)



## 7.3 - Cross sections

Bombard target nuclei with projectiles:



Definition of cross section:

(7.24)

$$\begin{array}{l} \# \text{ of reactions} \\ \text{per second and target nucleus} \end{array} = \sigma \times \begin{array}{l} \# \text{ of incoming projectiles} \\ \text{per second and cm}^2 \end{array}$$

or in symbols:  $\boxed{\text{rate} = \sigma j}$

with  $j$  as particle number current density.  
 $j = n v$  with particle number density  $n$ )

(7.25)

Units for cross section:

1 barn =  $10^{-24}$  cm<sup>2</sup> (= 100 fm<sup>2</sup> or about half the size (cross sectional area) of a uranium nucleus)

# Complication (1) Determine $\sigma$

The cross section is a measure of how likely a photon gets absorbed when an atom is bombarded with a flux of photons.

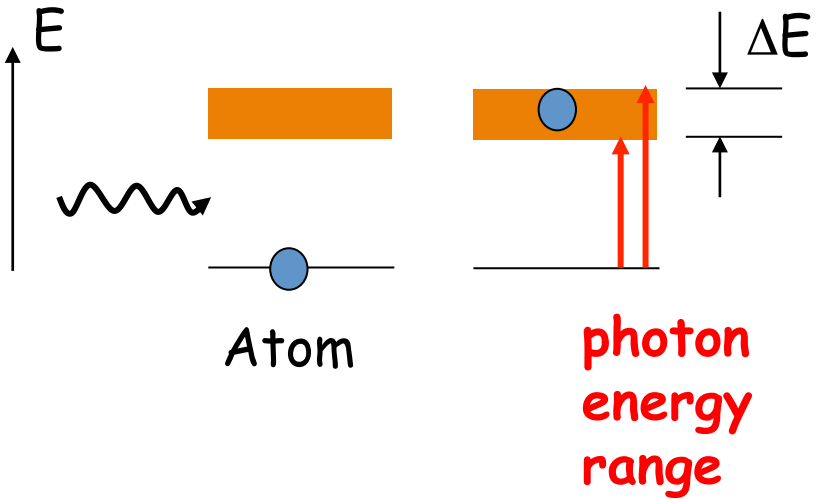
It depends on:

**Oscillator strength:** a quantum mechanical property of the atomic transition

Needs to be measured in the laboratory - not done with sufficient accuracy for a number of elements.

## Line width

The wider the line in wavelength, the more likely a photon is absorbed (as in a classical oscillator).



excited state has an energy width  $\Delta E$ . This leads to a range of photon energies that can be absorbed and to a line width

Heisenbergs uncertainty principle relates that to the **lifetime**  $\tau$  of the excited state

→ need lifetime of final state

$$\Delta E \cdot \tau = \hbar \quad (7.26)$$

# Determine $\sigma$

The lifetime of an atomic level in the stellar environment depends on:

**The natural lifetime** (natural width)

lifetime that level would have if atom is left undisturbed

**Frequency of Interactions of atom with other atoms or electrons**

**Collisions** with other atoms or electrons lead to deexcitation, and therefore to a shortening of the lifetime and a broadening of the line

Varying **electric fields** from neighboring ions vary level energies through **Stark Effect**

→ depends on **pressure**

→ need local **gravity**, or **mass/radius** of star

**Doppler broadening** through variations in atom velocity

• thermal motion → depends on **temperature**

• micro turbulence

**Need detailed and accurate model of stellar atmosphere !**

# Complication (2)

Atomic transitions depend on the **state of ionization** !

The number density  $n$  determined through absorption lines is therefore the number density of ions in the ionization state that corresponds to the respective transition.

To determine the total abundance of an atomic species one needs the fraction of atoms in the specific state of ionization.

Notation: I = neutral atom, II = one electron removed, III = two electrons removed .....

Example: a **CaII** line originates from singly ionized Calcium

## Frequently used notation:

- $n_i$  is the density of atoms in the  $i$ -th state of ionization, that is with  $i$  electrons removed.
- $g_i$  is the degeneracy of states for the  $i$ -ions
- $E_i$  is the energy required to remove  $i$  electrons from a neutral atom, creating an  $i$ -level ion.
- $n_e$  is the electron density

## 7.4 - The Saha Equation

In **Local Thermodynamical Equilibrium** (LTE) the distribution of atoms over various states of excitation and ionization is described by Boltzmann's relation for the relative populations of the ground atomic level  $n_1$  and the  $n^{\text{th}}$  level  $n_n$  (per unit volume),

$$\frac{n_1}{n_n} = \frac{g_1}{g_n} \exp\left(\frac{E_n - E_1}{kT}\right) \quad (7.27)$$

with corresponding energies  $E_1$  and  $E_n$  and statistical weights  $g_1$  and  $g_n$ :

Saha's equation is a relation, based on Boltzmann's distribution, for the ratio of number densities of two different ionization states of an atomic species. For free electrons and large enough temperatures, we will assume a Boltzmann distribution. The Saha equation follows as Saha's equation is a relation, based on Boltzmann's distribution, for the ratio of number densities of two different ionization states of an atomic species. For free electrons and large enough temperatures, we will assume a Boltzmann distribution. The **Saha equation** follows form

$$\frac{n_{i+1} n_e}{n_i} = \frac{g_{i+1} g_e}{g_i} \exp\left(-\frac{\chi_i}{kT}\right) \exp\left(-\frac{p^2}{2mkT}\right) d^3 p \quad (7.28)$$

# The Saha Equation

which, when integrated over the free electron momentum  $p$ , yields

$$\frac{n_{i+1} n_e}{n_i} = \frac{g_{i+1} g_e}{g_i} \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp\left( -\frac{\chi_i}{kT} \right) \quad (7.26)$$

where  $n_i$  is the number density of atoms in the  $i^{\text{th}}$  ionization state,  $n_{i+1}$  is the number density in the next ionization state,  $n_e$  is the number density of free electrons,  $\chi_i$  is the ionization energy difference between the  $i^{\text{th}}$  and  $i+1$  states, and the factors  $g_i$ ,  $g_{i+1}$ , and  $g_e$  are the corresponding statistical weights (degeneracy) of the discrete energy states. For free (unbound) electrons,  $g_e = 2$ , owing to two spin states. Generally, one can use  $g_n = 2n^2$  ( $n$  is the principal quantum number) for bound energy states. For the more complex atomic systems, a quantum treatment is usually necessary to calculate the statistical weights.

Let us see what we obtain for the ionization fraction of hydrogen in the solar photosphere, where  $E = 13.6 \text{ eV}$ ,  $T_{\text{ioniz}} = E/k \sim 160,000 \text{ K}$ ,  $T_{\text{gas}} \sim 6,000 \text{ K}$ , and  $n_{\text{H}} \sim 10^{17} \text{ cm}^{-3}$ . Thus, the actual gas temperature is much smaller than the temperature equivalent of the ionization energy. Naively one would expect a very low ionization fraction of hydrogen in the solar photosphere. Hydrogen is the dominating element, hence the electron density should be related to the hydrogen density by

# The Saha Equation

$$n_e = \frac{n_{i+1}}{n_i + n_{i+1}} n_H \quad (7.27)$$

The ratio of level partition sums is approximately 0.5, because the first excited level of atomic hydrogen is high compared to the thermal energy. This means that, approximately,  $n_i$  is the number of atoms in the ground state and  $n_{i+1}$  is the number of atoms either ionized or in the first excited state. Inserting the numbers in Saha's equation then gives the ionization fraction

$$\xi = \frac{n_{i+1}}{n_i + n_{i+1}} \rightarrow \frac{n_{i+1} n_e}{n_i n_H} = \frac{\xi^2}{1 - \xi} \sim 10^{-7} \rightarrow \xi \sim 3 \times 10^{-4} \quad (7.28)$$

where the intermediate step was obtained from Eq. (7.26) with  $\chi_i = 13.6$  eV. The result above seems small indeed, but repeating the calculation for a slightly hotter star with photospheric temperature  $T = 12,000$  K and the same gas density yields an ionization fraction  $\xi(T = 12,000 \text{ K}) \sim 0.3$ , much higher than what the ionization energy would suggest.

## 7.5 - Contributions to opacity

- Electron scattering (Thompson):  
where  $r_e$  is the classical electron radius.

$$\kappa = \frac{8\pi}{3} \frac{r_e^2}{m_H} \quad (7.29)$$

- Free-free transitions:  $\kappa_\nu \cong Z^2 \rho T^{-1/2} \nu^{-3}$  (7.30)

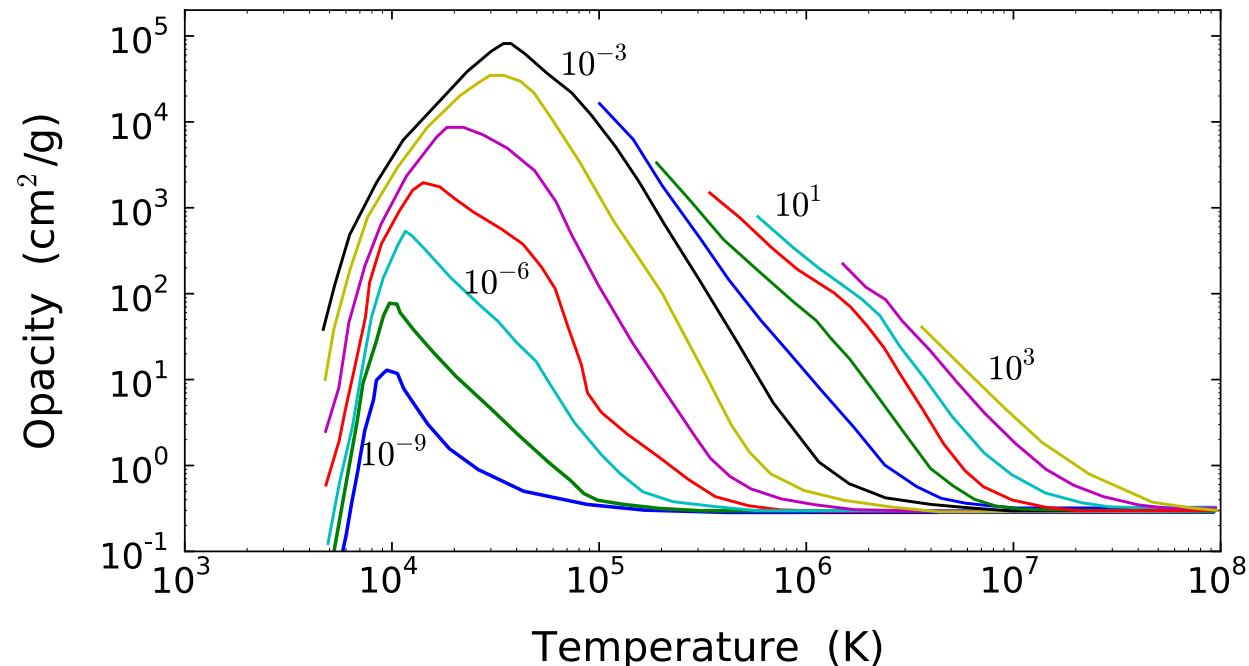
- Bound-free transitions

- Bound-bound transitions: contribute significantly for  $T < 10^6$  K (e.g., the Sun)

$$\kappa_\nu \cong \nu^3 \quad (7.31)$$

- + other atomic processes

Opacities are very complicated to calculate. The figure shows numerical calculations of opacities (in  $\text{cm}^2/\text{g}$ ) as a function of the temperature (in Kelvins) for several densities (in  $\text{g}/\text{cm}^3$ ).





## 7.6 - Solving the equations of stellar structure

Hence we now have four differential equations, which govern the structure of stars (note - in the absence of convection).

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$$\frac{dT(r)}{dr} = -\frac{3\rho(r)\bar{\kappa}(r)}{16\pi r^2 \sigma T(r)^3} L(r)$$

Information needs to be complemented with:

$P = P(\rho, T, \text{chemical composition})$

$\kappa = \kappa(\rho, T, \text{chemical composition})$

$\epsilon = \epsilon(\rho, T, \text{chemical composition})$

The equation of state

# Boundary conditions

Two of the boundary conditions are fairly obvious, at the center of the star  
 $M = 0, L = 0$  at  $r = 0$

At the surface of the star its not so clear, but we use approximations to allow solution.

There is no sharp edge to the star, but for the the Sun  $\rho(\text{surface}) \sim 10^{-4} \text{ kg m}^{-3}$ . This is much smaller than mean density  $\rho(\text{mean}) \sim 1.4 \times 10^3 \text{ kg m}^{-3}$  (which we derived).

We know the surface temperature ( $T_{\text{eff}} = 5780\text{K}$ ) is much smaller than its minimum mean temperature ( $2 \times 10^6 \text{ K}$ ).

Thus we make two approximations for the surface boundary conditions:

$$\rho = T = 0 \text{ at } r = r_s$$

i.e. that the star does have a sharp boundary with the surrounding vacuum.

# Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of  $M$  rather than  $r$ .

If we divide the other three equations by the equation of mass conservation, and invert the latter, we get

$$\frac{dr}{dM} = \frac{1}{4\pi r^2 \rho} \quad (7.32) \quad \frac{dL}{dM} = \varepsilon \quad (7.33)$$

$$\frac{dP}{dM} = -\frac{GM}{4\pi r^4} \quad (7.34) \quad \frac{dT}{dM} = -\frac{3\kappa_R L}{64\pi^2 r^4 acT^3} \quad (7.35)$$

With boundary conditions:  
 $r = 0, L = 0$  at  $M = 0$   
 $\rho = 0, T = 0$  at  $M = M_s$

We specify  $M_s$  and the chemical composition and now have a well defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

## Role of nuclear abundances

If there are no bulk motions in the interior of the star, then any changes of chemical composition are localised in the element of material in which the nuclear reactions occurred. So star would have a chemical composition which is a function of mass  $M$ .

In the case of no bulk motions - the set of equations we derived must be supplemented by equations describing the rate of change of *abundances* of the different chemical elements. Let  $C_{X,Y,Z}$  be the chemical composition of stellar material in terms of mass fractions of hydrogen ( $X$ ), helium, ( $Y$ ) and metals ( $Z$ ) [e.g. for solar system  $X = 0.7$ ,  $Y = 0.28$ ,  $Z = 0.02$ ]

$$\frac{\partial(C_{X,Y,Z})_M}{\partial t} = f(\rho, T, C_{X,Y,Z}) \quad (7.36)$$

Such a model evolves as

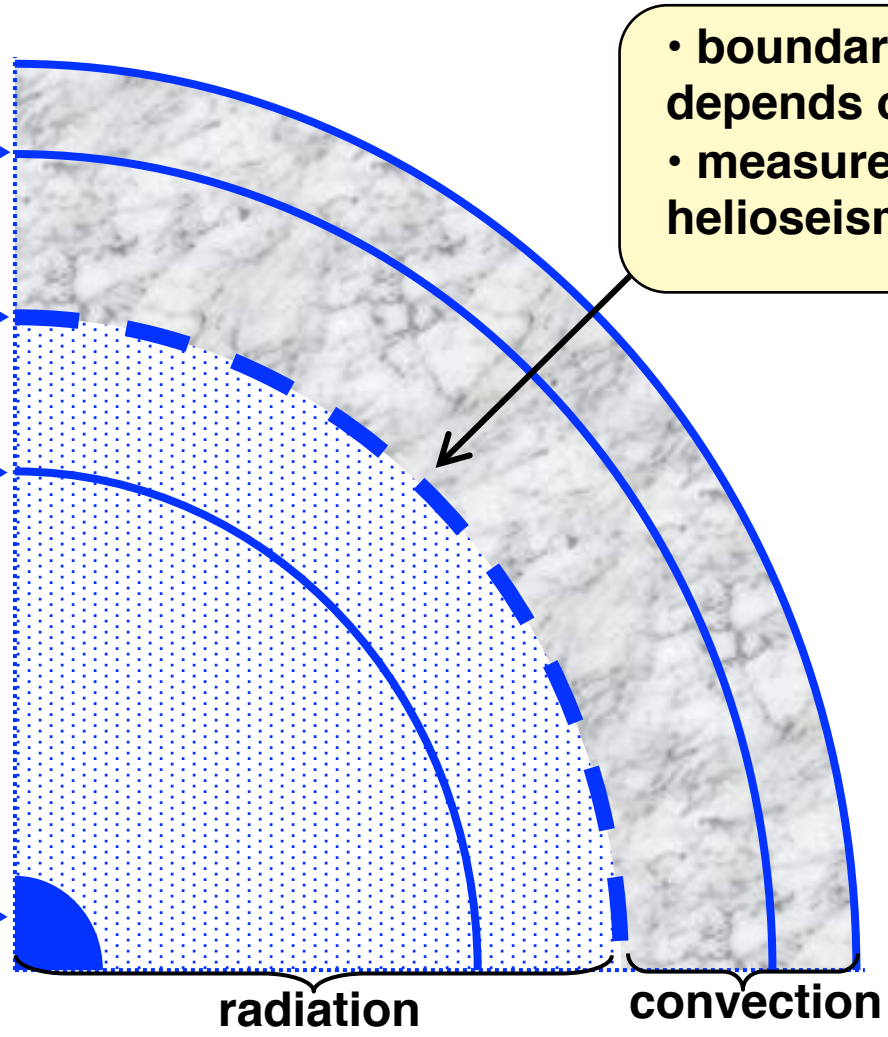
$$(C_{X,Y,Z})_{M,t_0+\delta t} = (C_{X,Y,Z})_{M,t_0} + \frac{\partial(C_{X,Y,Z})_M}{\partial t} \quad (7.37)$$

In solving the equations of stellar structure the eqns appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.

# Ex: - Solar model

T(eV)	$n_e$ (cm <sup>-3</sup> )	$r/R_\odot$
54	$1 \times 10^{22}$	0.90
182	$9 \times 10^{22}$	0.7133
293	$4 \times 10^{23}$	0.55
1360	$6 \times 10^{25}$	0

Solar model : J.N. Bahcall et al, Rev. Mod. Phys. 54, 767 (1982)



- boundary position depends on transport
- measured with helioseismology

Figure: Jim Bailey, Sandia

Transport depends on opacity, composition,  $n_e$ , T