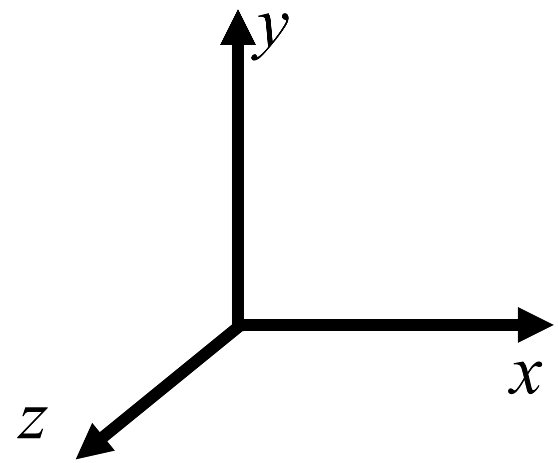
Appendix Lecture 1

Special Relativity
&
General Relativity

A.1 – Galileo's relativity

Newton's laws of motion must be implemented relative to some reference frame.

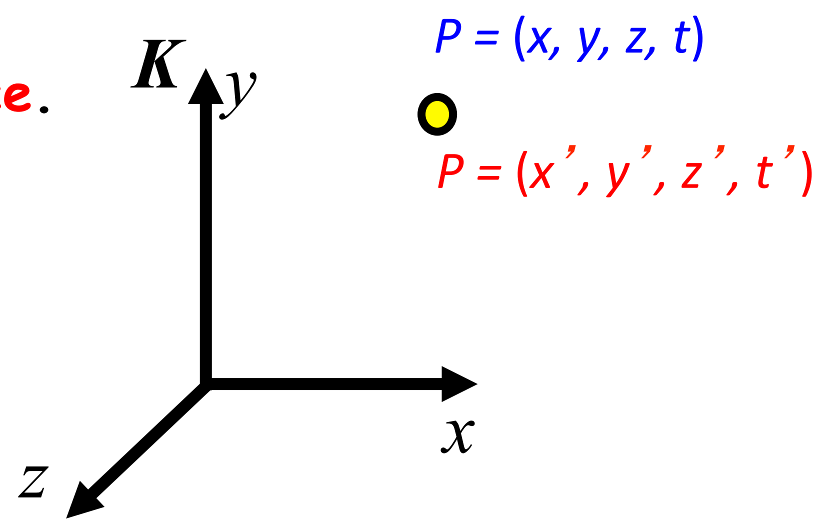
A reference frame is called an **inertial frame** if Newton's laws are valid in that frame.



Such a frame is established when a body, not subjected to net external forces, moves in rectilinear motion **at constant velocity**.

If Newton's laws are valid in one reference frame, then they are also valid in another reference frame moving at a uniform velocity relative to the first system.

This is referred to as the **Galilean invariance**.



Galilean transformation: for a point P

In one frame K: $P = (x, y, z, t)$

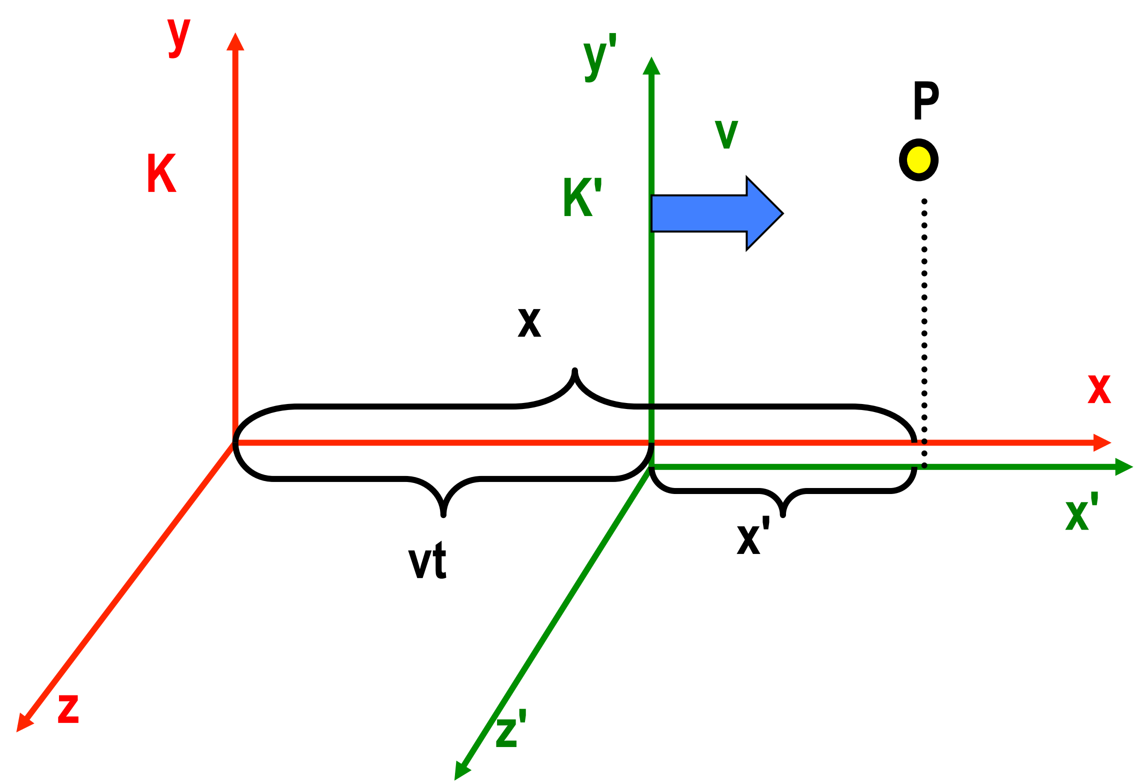
In another frame K: $P = (x', y', z', t')$

Galilean transformation

K' has a constant relative velocity (here in the *x*-direction) with respect to K.

Time (t) for all observers is a **fundamental invariant**, i.e., it is the same for all inertial observers.

$$x' = x - \mathrm{v}t$$
$$y' = y \qquad (A.1)$$
$$z' = z$$
$$t' = t$$



$$x = x' + \mathrm{v}t'$$
$$(A.2) \qquad y = y'$$
$$z = z'$$
$$t = t'$$

The inverse relations:

- replace -v with +v.

- Replace "primed" quantities with "unprimed" and "unprimed" with "primed."

Newton's laws of mechanics

Newton's laws of mechanics are in agreement with Galileo transformations

1. A body, not acted upon by any force, stays at rest or remains in uniform motion.
2. To change its velocity, we need a force

Force = mass x acceleration
(acceleration = change in velocity)

$$\mathbf{F} = m\mathbf{a} \equiv m\frac{\Delta \mathbf{v}}{\Delta t} \qquad (A.3)$$

Velocity and acceleration under Galielo transformations:

$$u_x \equiv \frac{\Delta x}{\Delta t} = \frac{\Delta(x' + vt)}{\Delta t} = \frac{\Delta x'}{\Delta t} + \frac{\Delta(vt)}{\Delta t} \equiv u_x' + v$$

$$u_y = u_y'; \qquad u_z = u_z'$$

velocity of an object in K is equal to its velocity in K', plus the velocity of K' with respect to K

$$a_y \equiv \frac{\Delta u_y}{\Delta t} = \frac{\Delta(u_y' + v)}{\Delta t} = \frac{\Delta u_y'}{\Delta t} + \underbrace{\frac{\Delta v}{\Delta t}}_{= 0 \text{ as } v = \text{const}} \equiv a_y'$$

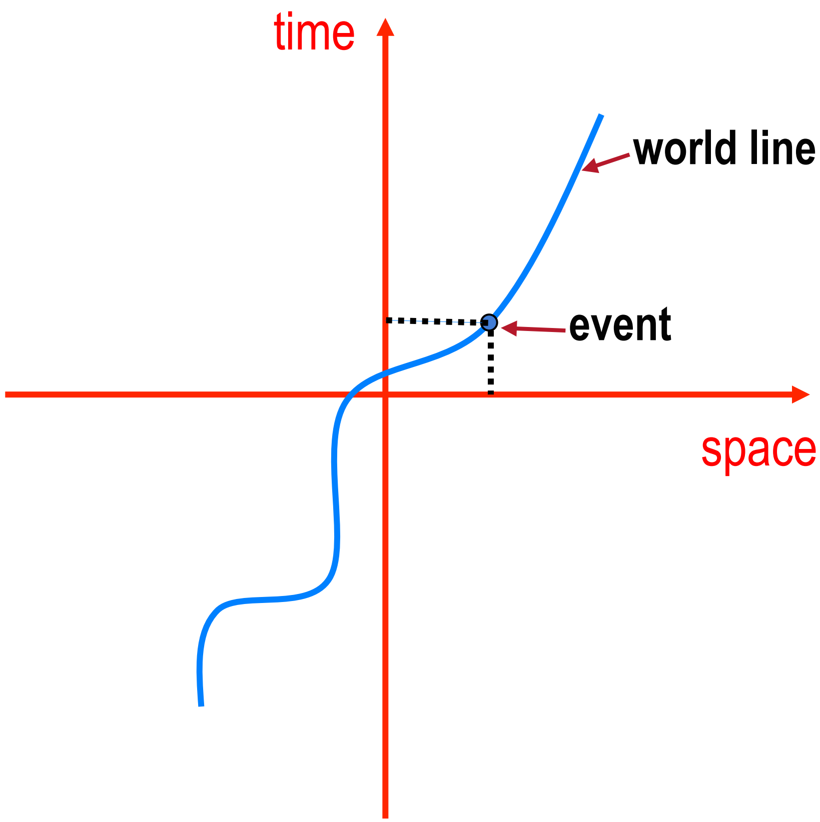
Accelerations are the same in both K and K' frames!

Newtonian forces will be the same in both frames

A.2 – Einstein's relativity

Einstein's relativity is based on two postulates:

- **All laws of nature are the same in all inertial frames**
  - This is the same as Galileo's relativity
- **The speed of light is independent of the motion of its source**
  - This simple statement requires a truly radical re-thinking about the nature of space and time!



In Newtonian physics, we assumed that $t' = t$.

With synchronized clocks, events in K and K' can be considered simultaneous.

Einstein realized that each system must have its own observers with their own synchronized clocks and meter sticks.

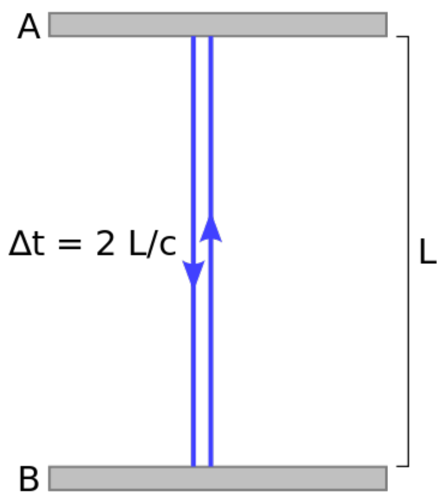
Events considered simultaneous in K may not be in K'.

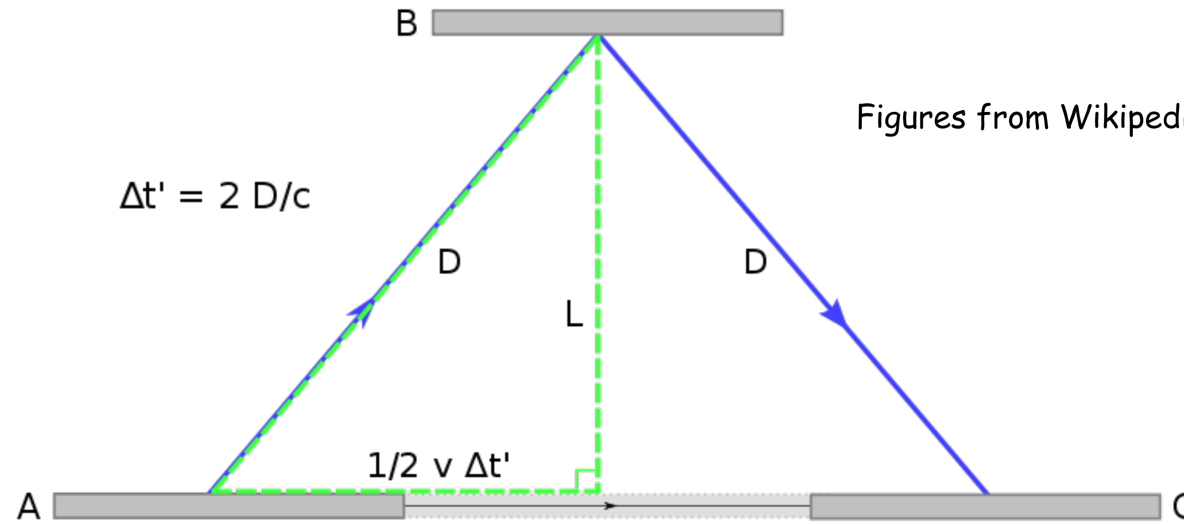
Time and space are inseparable. We need to describe a physics event in a **space-time diagram**.

Also, time may pass more slowly in some systems than in others.

Time dilatation

Time dilation formula is a consequence of Einstein's fundamental postulates. Let us consider a light clock. Ticks occur every time a light pulse is reflected back to the lower mirror





Figures from Wikipedia

Observer at rest sees time $2L/c$.

Observer moving parallel relative to setup, sees longer path, time > $2L/c$, same speed c.

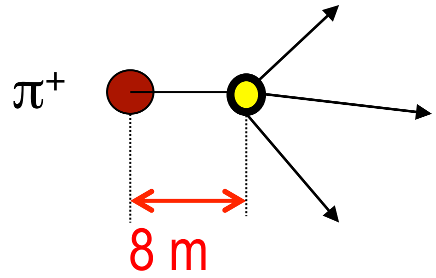
$$\Rightarrow \quad \Delta t' = 2\frac{D}{c} = \frac{2\sqrt{L^2 + (v\Delta t'/2)^2}}{c} = \frac{\sqrt{4L^2 + (v\Delta t')^2}}{c} = \sqrt{(\Delta t)^2 + \left(\frac{v}{c}\Delta t'\right)^2}$$

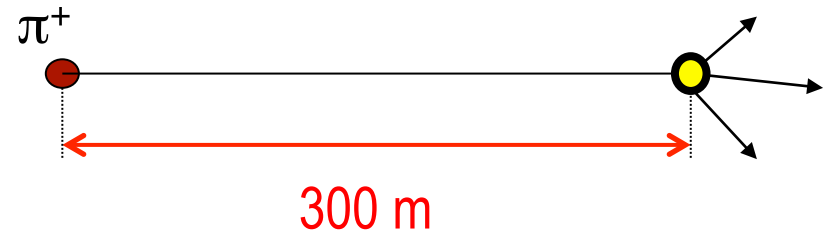
Solving for Δt', one gets

$$\boxed{\Delta t' = \Delta t \cdot \frac{1}{\sqrt{1 - v^2/c^2}}} \qquad (A.4)$$

Since $v/c < 1$ (always), then $\Delta t' > \Delta t$. That is, observer in motion sees the "event" (tick: light leaves lower part + "tack" light arrives back) taking a longer time than observer at rest. This is the **time dilatation** effect.

Time dilatation





No time dilation

With time dilation

Time in a moving system slows down compared to a stationary system!

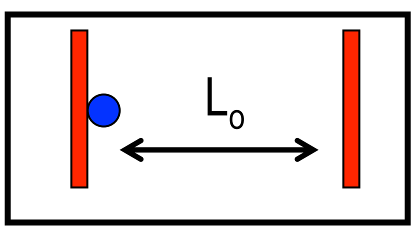
- E.g., **charged pions** have a lifetime of $t = 2.56 \times 10^{-8}$ s, so most of them would decay after traveling $ct = 8$ m.
- But we have no trouble seeing them produced at one point and detecting them arriving hundreds of meters away.

Space contraction

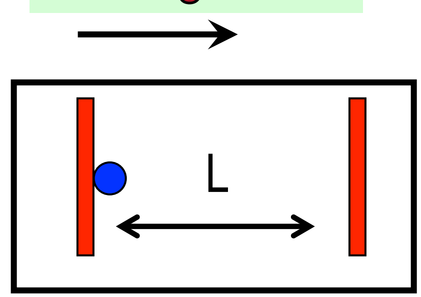
Consider the time for a pulse parallel to the system velocity to do a round trip:

Stationary clock:

$$t_0 = 2\frac{L_0}{c}$$



Moving clock:



$$t = \frac{L}{c-v} + \frac{L}{c+v}$$

$$\boxed{L = L_0 \cdot \sqrt{1 - v^2/c^2}} \quad (A.5)$$

An observer moving along an object will find it shorter than it would be if the observer was standing still.

Transformation between systems

The constancy of the speed of light is not compatible with Galilean transformations. Consider a wave front starting at the origin of two frames whose origin coincide at $t = 0$. In terms of the coordinates of the two frames

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

According to the Galilean transformation

$$x' = x - vt$$

$$y' = y, \quad z' = z, \quad t' = t$$

$$\Rightarrow \quad x'^2 + y'^2 + z'^2 = (x^2 - 2xvt + v^2 t^2) + y^2 + z^2 \neq c^2 t^2 \qquad (A.6)$$

Therefore the Galilean transformation is not compatible with the constancy of the speed of light

There are a couple of extra terms ($-2xvt + v^2 t^2$) in the primed frame.

Transformation between systems

Let us try a possible solution: $x' = \gamma(x - vt)$ and $x = \gamma'(x' + vt')$, where γ could be anything.

By Einstein's first postulate: $\gamma' = \gamma$

The wave-front along the x'- and x-axes must satisfy: $x' = ct'$ and $x = ct$

Thus $ct' = \gamma(ct - vt)$ or $t' = \gamma t(1 - v/c)$

Thus $ct' = \gamma(ct - vt)$ or $t' = \gamma t(1 - v/c)$ and $ct = \gamma'(ct' + vt')$ or $t = \gamma' t'(1 + v/c)$

Combining the two solutions for t and t' we get

$$t' = \gamma^2 t' \left(1 - \frac{\mathrm{v}}{c}\right)\left(1 + \frac{\mathrm{v}}{c}\right)$$

or

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad (A.7)$$

γ is also know as **Lorentz factor**.

We also get a relation for the time and position x, t and x', t' in K and K'

$$x' = \gamma(x - \mathrm{v}t) \qquad (A.8)$$

$$t' = \gamma(t - \mathrm{v}x/c^2) \qquad (A.9)$$

A.3 – Lorentz transformations

What we have accomplished in the previous slides it to show how position and time relate under a transformation between systems moving with a velocity v relative to each other. Such transformations are called by Lorentz transfomations. They obey Einstein's condition that the speed of light is the same in all systems. Summarizing, the Lorentz transformations are:

$$x' = \frac{x - \mathrm{v}t}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \mathrm{v}x/c^2}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

$$(A.10)$$

$$\beta = \mathrm{v}/c$$

$$\gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2/c^2}}$$

$$x' = \gamma\left(x - \beta ct\right)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \beta x/c\right)$$

$$(A.11)$$

Addition of velocities

Taking differentials of the Lorentz transformation [here between the rest frame (K) and the moving frame (K')], we can compute the shuttle velocity in the rest frame ($u_x = dx/dt$):

$$\begin{aligned} dx &= \gamma(dx' + \mathrm{v}\,dt') \\ dy &= dy' \\ dz &= dz' \\ dt &= \gamma[dt' + (\mathrm{v}/c^2)dx'] \end{aligned} \qquad (A.13)$$

Defining velocities as: $u_x = dx/dt$, $u_y = dy/dt$, $u'_x = dx'/dt'$, etc., we find:

$$u_x = \frac{dx}{dt} = \frac{\gamma(dx' + \mathrm{v}\,dt')}{\gamma[dt' + (\mathrm{v}/c^2)\,dx']} = \frac{u'_x + \mathrm{v}}{1 + u'_x \mathrm{v}/c^2} \qquad (A.14)$$

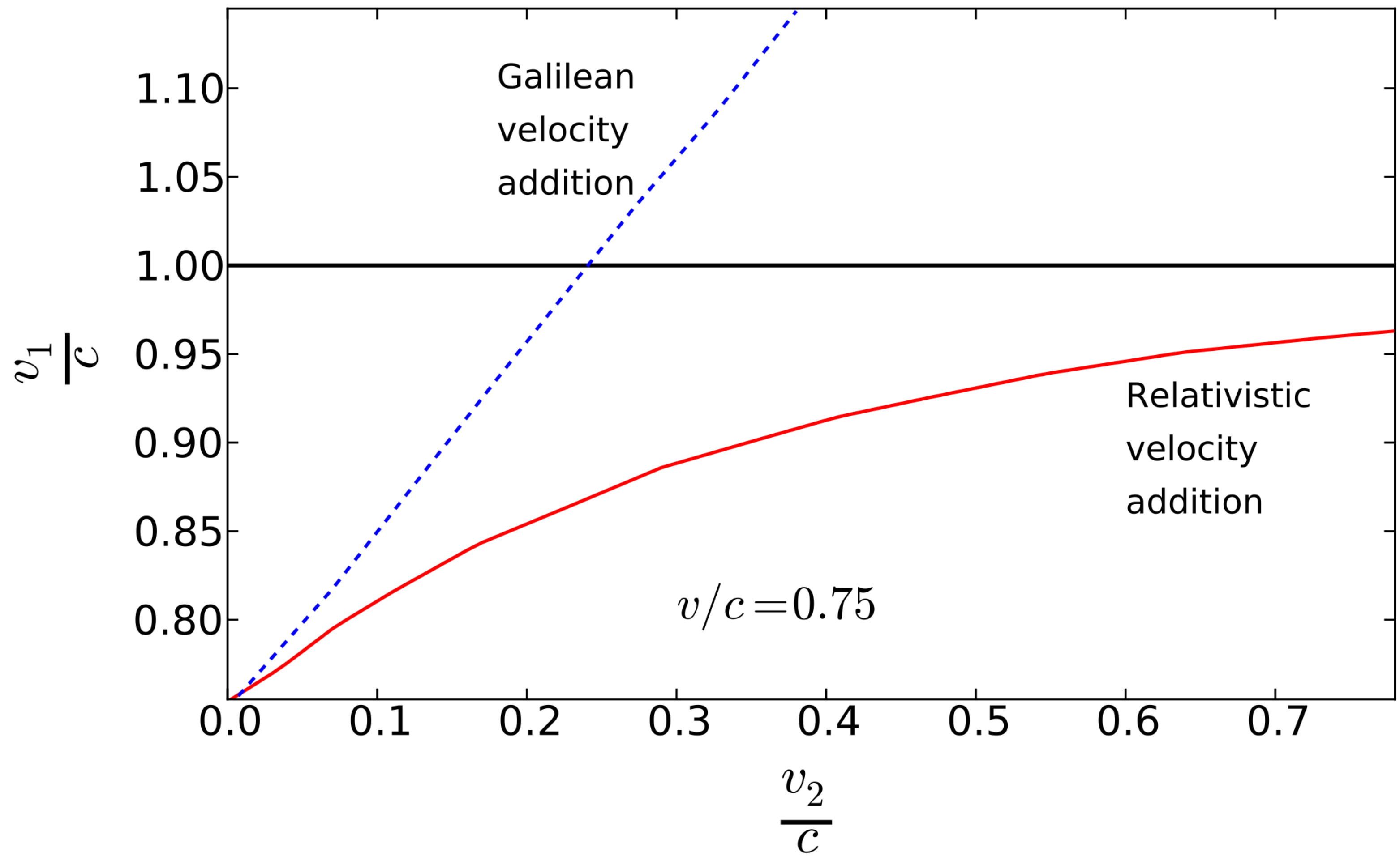
with similar relations for u_y and u_z.

$$u_y = \frac{dy}{dt} = \frac{dy'}{\gamma[dt' + (\mathrm{v}/c^2)\,dx']} = \frac{u'_y}{\gamma(1 + u'_x \mathrm{v}/c^2)} \qquad (A.15)$$

$$u_z = \frac{dz}{dt} = \frac{dz'}{\gamma[dt' + (\mathrm{v}/c^2)\,dx']} = \frac{u'_z}{\gamma(1 + u'_x \mathrm{v}/c^2)} \qquad (A.16)$$

Note the γ's in u_y and u_z.

Addition of velocities

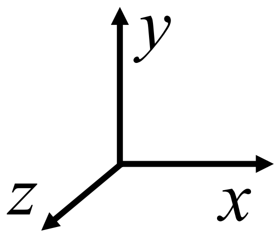


The figure highlights the difference between Galilean velocity addition, $v_2 = v_1 + v$, of velocities v_1 and v_2 measured in two different frames moving relative to each other with velocity v, and the relativistic velocity addition,
$v_2 = (v_1 + v)/(1 + v_1v/c^2)$.

One notices that the addition of velocities can never be larger than the speed of light.

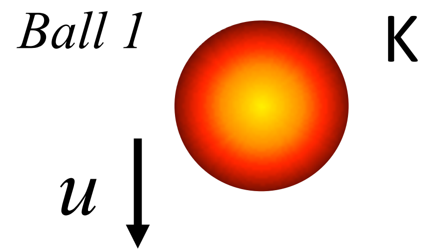
A.4 – Relativistic momentum and energy

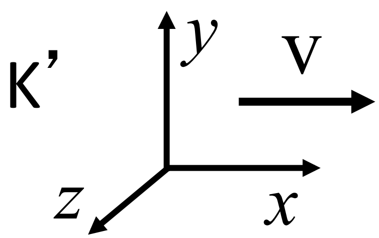
A ball (1) of mass *m* is thrown down in the minus *y*-direction in frame K. In the moving system K' another ball (2) is thrown upwards.



The momentum of the ball 1 thrown in K is entirely in the *y* direction

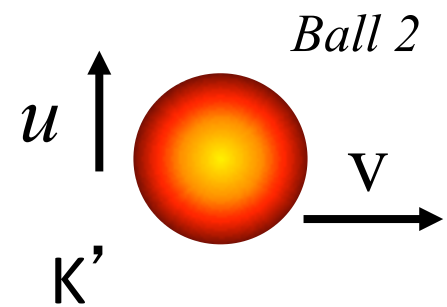
$$\mathrm{p}_{1y} = -\ \mathrm{mu}$$





If the two balls collide elastically, the change of *y*-momentum as observed in K for the ball thrown down is

$$\Delta \mathrm{p}_{2y} = 2\mathrm{mu}$$



In frame K' one measures the initial velocity of the thrown up ball to be: $u'_{2x} = 0$ and $u'_{2y} = u$.

In order to determine the velocity of ball 2, as measured in K, we use the relativistic velocity transformation equations:

$$u_x = \frac{\cancel{u'_x} + \mathrm{v}}{1 + \cancel{u'_x}\mathrm{v}/c^2} \qquad u_y = \frac{u'_y}{\gamma(1 + \cancel{u'_x}\mathrm{v}/c^2)}$$

Relativistic momentum

Thus, the velocity of ball 2, as measured in K, is

$$u_{2x} = \mathrm{v}$$

$$u_{2y} = \frac{u}{\gamma} = u\sqrt{1-\mathrm{v}^2/c^2}$$

Before the collision, the momentum of ball 2, as measured in K, is

$$\mathrm{p}_{2\mathrm{x}} = \mathrm{mv}$$

$$\mathrm{p}_{2\mathrm{y}} = \mathrm{mu}\sqrt{1-\mathrm{v}^2/c^2}$$

For a perfectly elastic collision, the momentum after the collision is

$$\Delta \mathrm{p}_{2\mathrm{y}} = -2\,\mathrm{m}\,\mathrm{u}\sqrt{1-\mathrm{v}^2/c^2}$$

The change in *y*-momentum of ball 2 according to frame K is

$$\Delta \mathrm{p}_{2\mathrm{y}} = -2\,\mathrm{m}\,\mathrm{u}\sqrt{1-\mathrm{v}^2/c^2}$$

whose magnitude is **different** from that of ball 1: $\mathrm{p}_{1\mathrm{y}} = -\,\mathrm{mu}$

Conservation of linear momentum requires the total change in momentum of the collision, $\Delta p_1 + \Delta p_2$, to be zero. **The addition of these y-momenta is clearly not zero.**

Relativistic momentum

Linear momentum is not conserved if we use the conventions for momentum from classical physics — even if we use the velocity transformation equations from special relativity.

There is no problem with the *x* direction, but there is a problem with along the direction the ball is thrown in each system, the *y* direction.

Rather than abandon the conservation of linear momentum, we can make a modification of the definition of linear momentum that preserves both momentum conservation and Newton's second law.

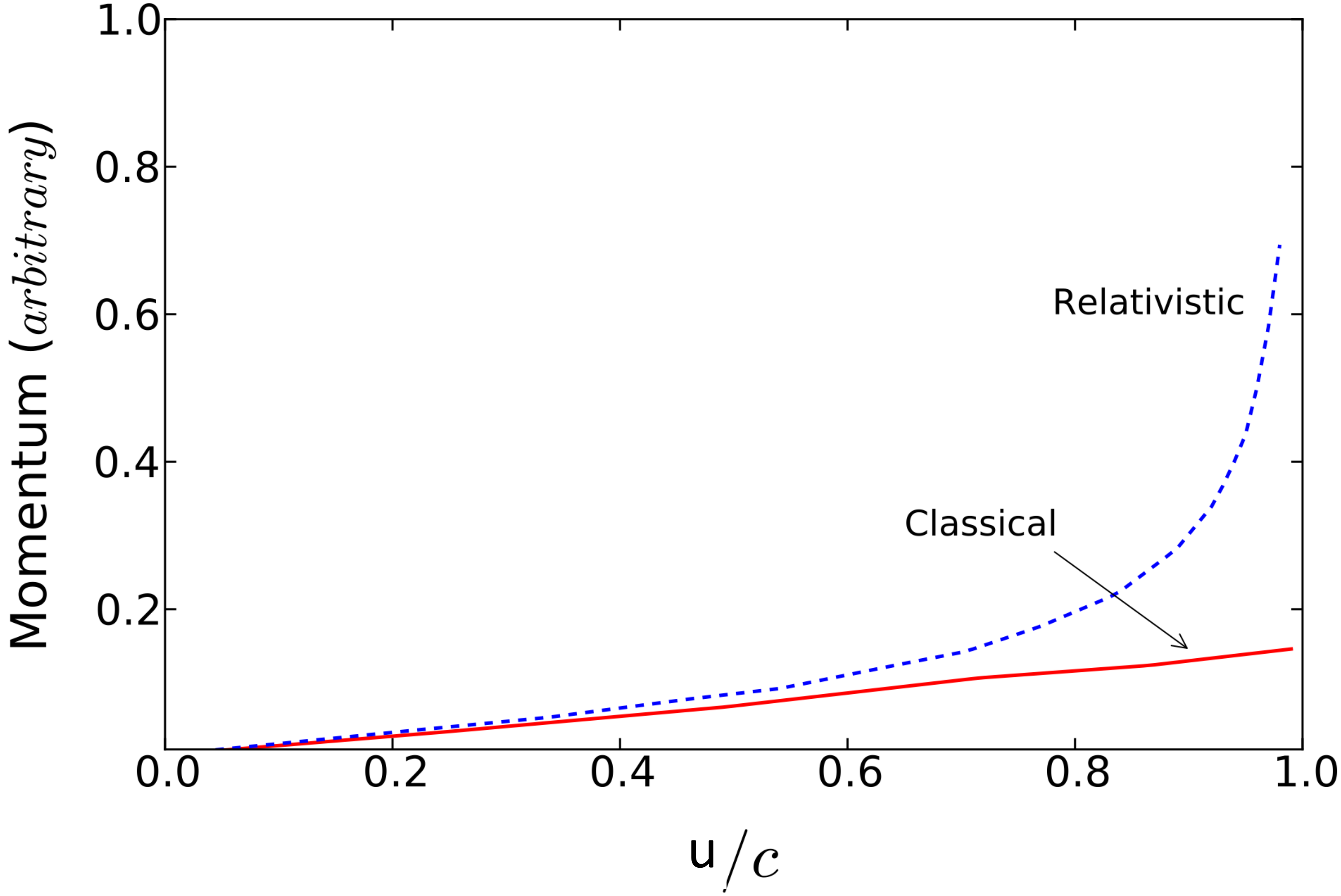
To do so requires re-examining momentum to conclude that

$$\boxed{\mathrm{p} = \gamma \mathrm{m} \mathrm{u}} \qquad (A.17)$$

$$\boxed{\gamma = \frac{1}{\sqrt{1 - \mathrm{u}^2 / c^2}}} \qquad (A.18)$$

Using this definition, one can show that momentum conservation is hold. I will leave that as an exercise.

Relativistic momentum



The figure highlights the difference between the classical momentum $p = mu$ and the relativistic momentum $p = \gamma mu$.

One notices that the relativistic momentum increases very fast as the speed of mass m approaches the speed of light.

Relativistic Energy

Let us modify Newton's second law to include the new definition of linear momentum. The force becomes (just consider motion in one dimension)

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma mu) = \frac{d}{dt}\left(\frac{mu}{\sqrt{1-u^2/c^2}}\right)$$

The differential work done to move a mass by a distance *x* is

$$dW = Fdx = \frac{dp}{dt}dx$$

Dividing by *dt*, we get

$$\frac{dW}{dt} = \frac{dp}{dt}\frac{dx}{dt} = \frac{dp}{dt}u$$

or, in terms of velocity derivatives

$$\frac{dW}{du}\frac{du}{dt} = \frac{dp}{du}\frac{du}{dt}u$$

$$\frac{dW}{du} = \frac{dp}{du}u$$

or,

$$dW = \frac{dp}{du}u\,du$$

Relativistic Kinetic Energy

The kinetic energy will be equal to the work done starting with zero energy and ending with W_0, or from zero velocity to u

$$K = \int_0^{W_0} dW = \int_0^u \frac{dp}{du'} u' du'$$

Integrating by parts

$$K = \int_0^u \frac{dp}{du'} u' du' = pu'\Big|_0^u - \int_0^u p\, du'$$

$$= pu - m\int_0^u \frac{u'}{\sqrt{1-u'^2/c^2}} du' = pu + mc^2 \sqrt{1-u'^2/c^2}\Big|_0^u$$

$$= \left(m\frac{u}{\sqrt{1-u^2/c^2}} \right) u + mc^2 \left(\sqrt{1-u^2/c^2} - 1 \right)$$

$$= mc^2 \left(\frac{u^2/c^2}{\sqrt{1-u^2/c^2}} + \frac{1-u^2/c^2}{\sqrt{1-u^2/c^2}} - 1 \right)$$

or, $$\boxed{K = mc^2 (\gamma - 1)} \qquad (A.19)$$

This is the **kinetic energy** in special relativity.

Relativistic Kinetic Energy

If velocity *u* is very small comparing to *c*, then the Lorentz factor becomes

$$\gamma = (1 - u^2/c^2)^{-1/2} = \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots\right)$$

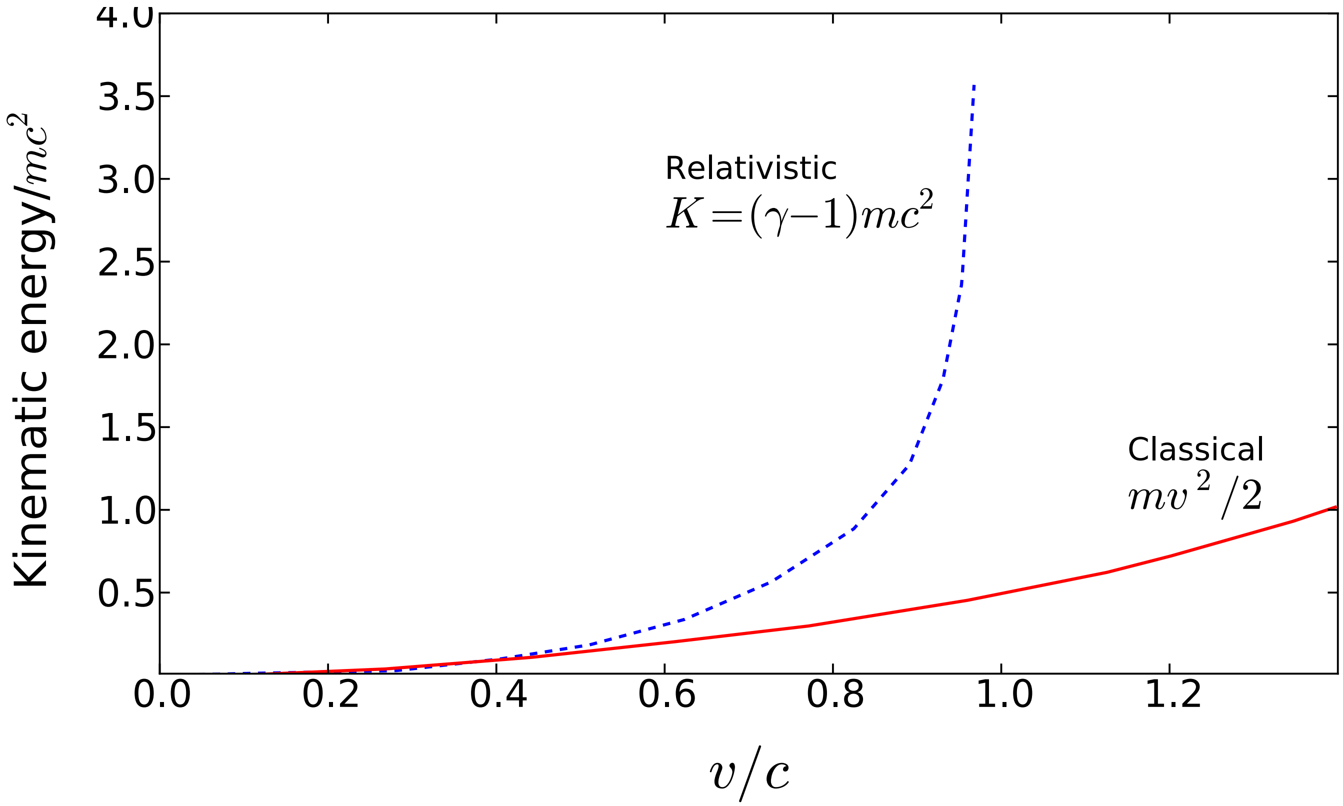
Thus, at lower velocities ($u \ll c$) the kinetic energy becomes

$$K \approx mc^2\left[\left(1 + \tfrac{1}{2}u^2/c^2\right) - 1\right] = \tfrac{1}{2}mu^2 \qquad (A.20)$$

This is exactly as we expected from Newtonian mechanics.

The figure highlights the difference between the classical kinetic energy $K = mu^2/2$ and the relativistic kinetic energy $K = mc^2(\gamma-1)$.

Even an infinite amount of energy is not enough to achieve c.



Relativistic Total Energy

The term mc^2 is called the **Rest Energy** :

$$E_0 = mc^2 \qquad (A.21)$$

The sum of the kinetic and rest energies is the total energy of the particle E and is given by

$$E = K + E_0 = \gamma mc^2 \qquad (A.22)$$

Since

$$\gamma^2 - 1 = \frac{u^2/c^2}{1-u^2/c^2} = \gamma^2 u^2/c^2$$

we get from Eq. (A.17)

$$m^2c^4\left(\gamma^2 - 1\right) = \gamma^2 m^2 u^2 c^2$$

or

$$E^2 = p^2c^2 + m^2c^4 \qquad (A.23)$$

This equation relates the total energy of a particle with its momentum. The quantities (E^2 - p^2c^2) and m are invariant quantities, i.e. the same in K and K'.

When a particle's velocity is zero (no momentum), Eq. (A.23) gives

$$E(u=0) = E_0 = mc^2 \qquad (A.24)$$

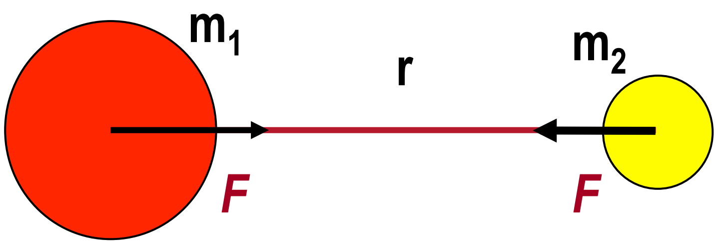
That is, relativity predicts that mass also contains energy. If energy is conserved, mass can be transformed into other forms of energy.

Eq. (A.24) is the world most famous equation.

General Relativity

A.5 - General Relativity

Newton's law of gravitation



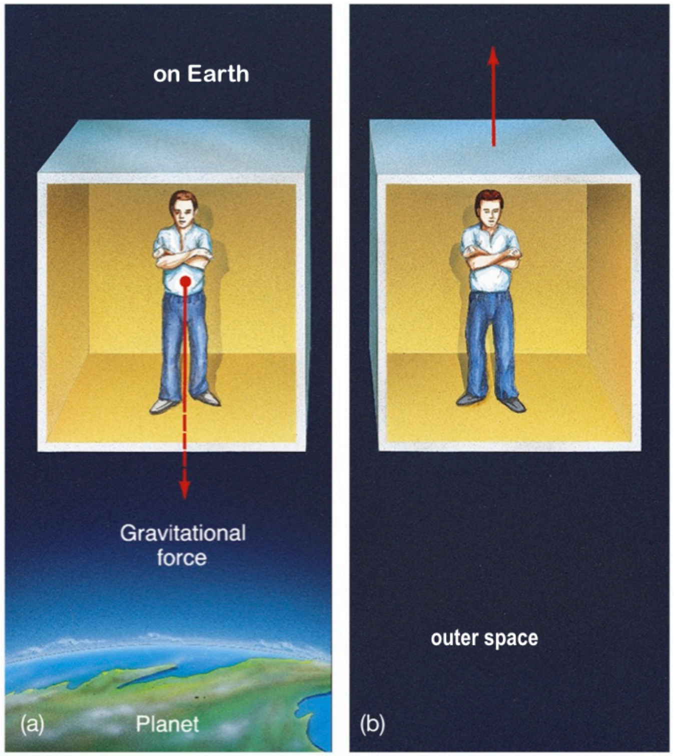
$$\boxed{F = G\frac{m_1 m_2}{r^2}} \qquad (A.25)$$

We know there are **4 forces of nature**:

- **Gravity, Electromagnetism, Weak & Strong Nuclear** forces
- Gravity is by far the weakest force, but it is also the most obvious: it's universal, acting the same on all forms of matter

Einstein realized there that is an equivalence between gravity and acceleration: you are weightless in a plummeting elevator. This is the **equivalence principle.**

Another form of **Einstein's equivalence principle**: an observer inside an enclosed box cannot tell the difference between being at rest on Earth's surface (a) or being accelerated in outer space (b).



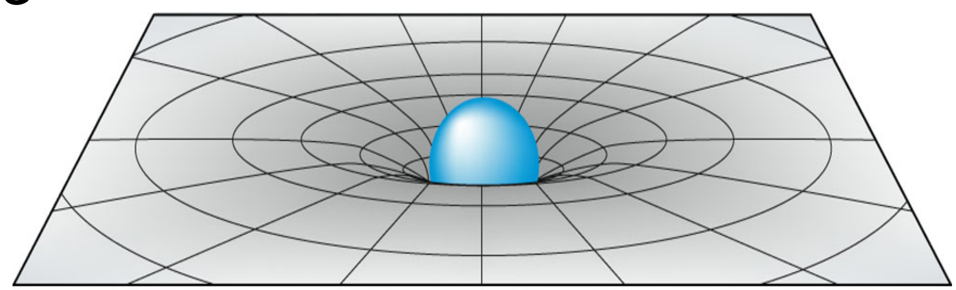
General Relativity

In 1916 Einstein published the final form of the **General Theory of Relativity.**

We can think of gravity as a feature of the background in which we live. This background is space and time: **spacetime**

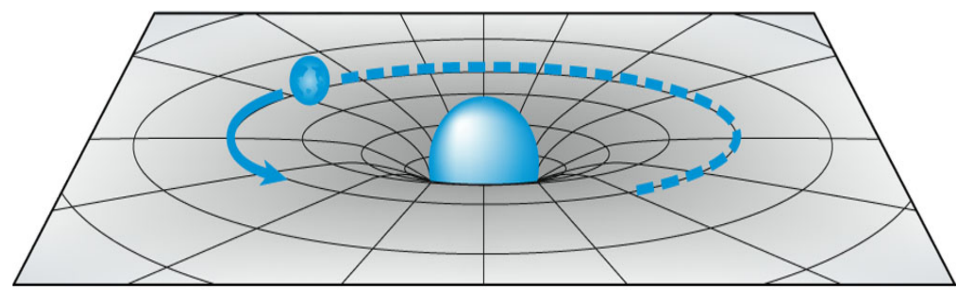
What we experience as gravity is actually **the curvature of spacetime**

In general relativity (GR), matter warps space-time, so that the straightest and shortest path *(**geodesic**)* looks like a curve to us.



Mass tells space how to curve.

Space tells matter how to move.



The figure shows an analogy: weight on a tight rubber sheet depresses it (a) , so a ball is deflected around it (b). That is how GR describes the motion of a planet around the Sun, and not by means of a force, as implied by Newton's gravitational force, Eq. (A.25). However, Einstein showed us that Newton's law is a limit of GR for small masses.

General Relativity

We know how to describe motion of objects exactly (remember rocket science) using Newton's gravitational law. There must be a way describe exactly the motion without forces, according to GR.

Well, it is complicated.

I will give a very short tour of GR next.

The **space-time interval**, Δs defined as

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2 \qquad (A.26)$$

is Lorentz invariant. That is, if we use the Lorentz transformations with Eq. (A. 13), with $\Delta x = dx$, $\Delta t = dt$, etc., we get $\Delta s'^2 = \Delta s^2$.

This interval can be written in terms of the **space-time metric**

$$\Delta s^2 = \begin{bmatrix} c\Delta t & \Delta x & \Delta y & \Delta z \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} \qquad (A.27)$$

The space-time metric

We can rewrite the expression for the space-time interval

$$\Delta s^2 = \sum_{\mu=0,\ldots,3} \sum_{\nu=0,\ldots,3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \qquad (A.28)$$

where

$$x^0 = ct \qquad x^1 = x$$
$$x^2 = y \qquad x^3 = z$$

$$(A.29)$$

and

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \eta_{\mu\nu} = & \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \end{array}$$

$$(A.30)$$

It is economical to use the **Summation Notation**: the summed indices occur once as **subscripts** and again as **superscripts**:

$$\Delta s^2 = \sum_{\mu=0,\ldots,3} \sum_{\nu=0,\ldots,3} \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

$$(A.31)$$

Use of Greek letters means ν = 0, 1, 2, 3

$$\Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \qquad (A.32)$$

When the same index appears as a superscript and a subscript, summation is assumed, and we can omit the summation symbols.

Tensors

In General Relativity, space is curved, and the space-time metric can be more complex. The more general metric coefficients of general relativity (which may not be -1's, 0's, and 1's) are denoted by $g_{\mu\nu}$:

$$\Delta s^2 = g_{\mu\nu}\, \Delta x^\mu \Delta x^\nu \qquad (A.33)$$

Example: An expanding (flat) universe

$$\Delta s^2 = a(t)^2 \left[\Delta x^2 + \Delta y^2 + \Delta z^2\right] - c^2 \Delta t^2 \qquad (A.34)$$

$$\text{where } a(t) \sim t^{\,q}$$

Values of q range from *1/2* (in a radiation-dominated universe) to *2/3* (in a matter-dominated universe). **(See Lecture 4).**

$g_{\mu\nu}$ is an tensor. A **tensor** is a function of one or more vectors that yields a real number. $g_{\mu\nu}$ takes two input vectors and yields a number: the interval Δs^2.

Because $g_{\mu\nu}$ operates on two vectors, we say it's a tensor of **rank 2**.

Example: a **vector** can undergo dot products with other vectors to yield a number, so **it's a tensor of rank 1**. **Scalars have rank zero**.

The rank is also the number of indices on the tensor and the dimension of the matrix necessary to write it down.

Geodesics

GR distinguishes between vectors and tensors that are **covariant** (with lower indices) and **contravariant** (with upper indices). To raise or lower an index, simply multiply by the metric:

$$x_\mu = g_{\mu\nu} x^\nu \qquad (A.35)$$

$$\Gamma^\alpha_{\ \mu\kappa} = g_{\mu\nu} \Gamma^{\alpha\nu}_{\ \ \ \kappa} \qquad (A.36)$$

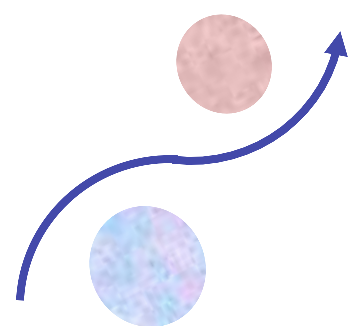
Ordinarily, we don't usually have to worry about this because our metric is simple, and covariant and contravariant tensors are essentially the same.

To raise the indices of the metric $g_{\mu\nu}$ itself, just take its inverse

$$g^{\mu\nu} = \left[g\right]^{-1}_{\mu\nu} \qquad (A.37)$$

In Newtonian space, **geodesics** are straight lines, and one way of saying this is that acceleration is zero

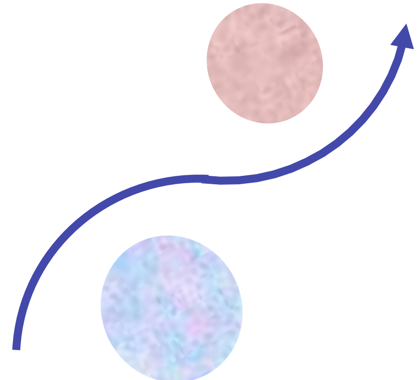
$$\frac{d^2 x^\alpha}{d\tau^2} = 0 \qquad (A.38)$$



where τ is **proper time** (i.e., the time measured in the frame of reference of the particle) , and x^a is the position vs. τ of the particle.

Curved Spaces

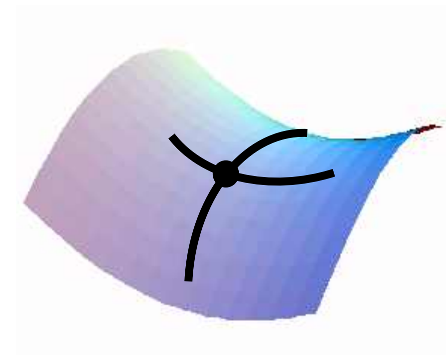
In curved space, this expression generalizes to



$$\frac{d^2 x^\alpha}{d\tau^2} = -\Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\tau} \frac{dx^\gamma}{d\tau} \qquad (A.39)$$

where $\Gamma^\alpha_{\beta\gamma}$ is called a **Christoffel symbol**, given by

$$g_{\alpha\delta}\Gamma^\delta_{\beta\gamma} = \frac{1}{2}\left(\frac{dg_{\alpha\beta}}{dx^\gamma} + \frac{dg_{\alpha\gamma}}{dx^\beta} - \frac{dg_{\beta\gamma}}{dx^\alpha} \right) \qquad (A.40)$$



The curvature of space-time is complicated because there are several dimensions, and the curvature at each point can be different in each dimension (including time). Think of a saddle in two dimensions for which the curvature depends on the direction.

The curvature of space-time is given by the **Ricci Tensor**

$$R_{\alpha\beta} = \frac{d\Gamma^\gamma_{\alpha\beta}}{dx^\gamma} - \frac{d\Gamma^\gamma_{\alpha\gamma}}{dx^\beta} + \Gamma^\gamma_{\alpha\beta}\Gamma^\delta_{\gamma\delta} - \Gamma^\gamma_{\alpha\delta}\Gamma^\delta_{\beta\gamma} \qquad (A.41)$$

Einstein Tensor

The **Einstein tensor** can be written in terms of the Ricci tensor as

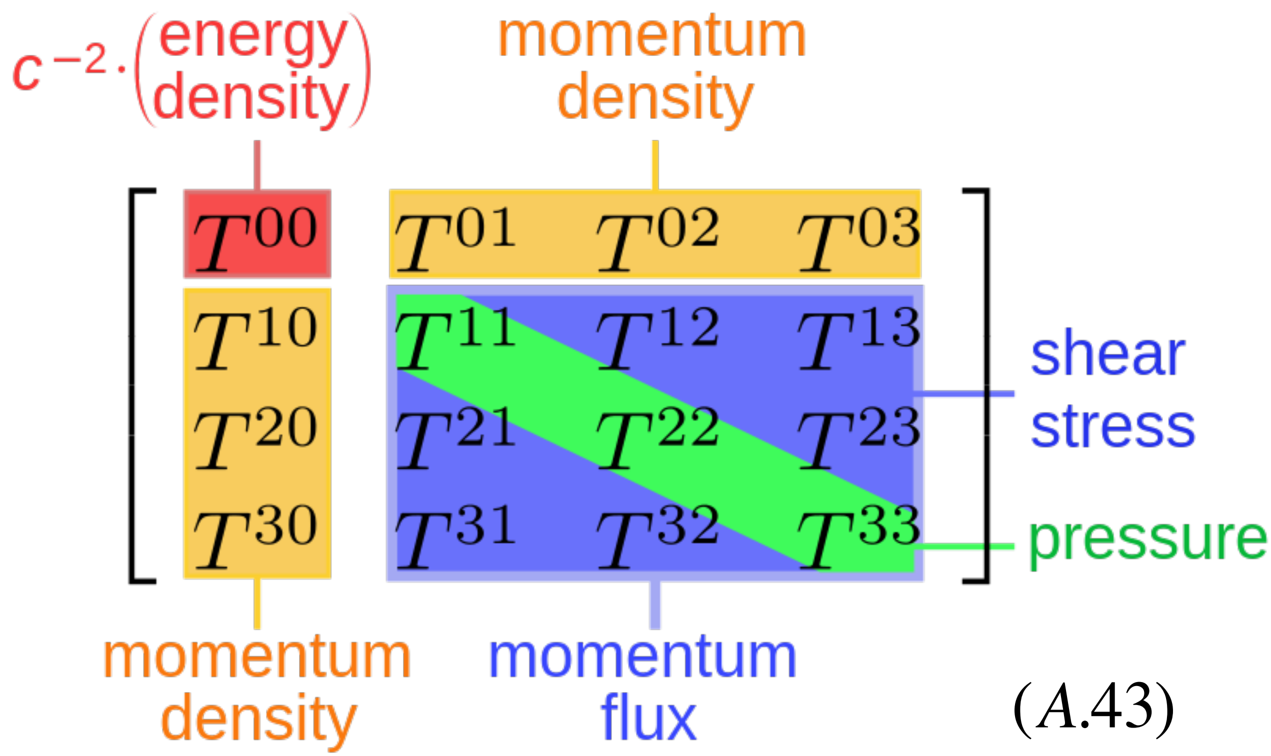
$$G_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} \qquad (A.42)$$

where R is the trace (i.e., the sum $R_{\mu\mu}$) of the Ricci tensor

Matter's effect on space-time occurs through the **stress-energy tensor**, T.

$T_{00} = T_{tt}$ is the **mass-energy density**

$T_{10} = T_{xt}$, $T_{20} = T_{yt}$ and $T_{30} = T_{zt}$ are how fast the matter is moving — its **momentum**



$(A.43)$

$T_{11} = T_{xx}$, $T_{22} = T_{yy}$ and $T_{33} = T_{zz}$ are the **pressures** in each of the three directions $T_{12} = T_{xy}$, $T_{13} = T_{xz}$ and $T_{23} = T_{yz}$ are the **stresses in the matter**.

Einstein Field Equations

The following set of coupled nonlinear partial differential equations (one for each element) relates the curvature of space, $G_{\mu\nu}$, to the energy-momentum tensor, $T_{\mu\nu}$:

$$G_{\mu\nu} = \frac{8\pi}{c^4} G\, T_{\mu\nu} \qquad (A.44)$$

Only six component equations are independent.

where G is the usual gravitational constant.

The goal is to solve for $g_{\mu\nu}$, for all values of μ and ν. In free space, where $T_{\mu\nu} = 0$, this reduces to

$$R_{\mu\nu} = 0 \qquad (A.45)$$

One can show that Einstein's Field Equations reduce to Newton's law of gravity in the weak-field and slow-motion limit.

As mentioned in Lecture 3, Einstein introduced the **Cosmological constant** by modifying his equation to

$$G_{\mu\nu} + g_{\mu\nu}\Lambda = \frac{8\pi}{c^4} G\, T_{\mu\nu} \qquad (A.46)$$

Ex: The Schwarzschild Solution

Using spherical coordinates, ρ, θ, ϕ, and spherical symmetry, we can solve Einstein's Field Equations (with Λ = 0) for the metric to find

$$\Delta s^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} \Delta r^2 + r^2 \Delta\Omega^2 - \left(1 - \frac{2GM}{rc^2}\right) \Delta t^2 \qquad (A.47)$$

The other elements of $g_{\mu\nu}$ are zero, and $\Delta\Omega^2 = \Delta\theta^2 + \sin^2\theta\,\Delta\phi^2$

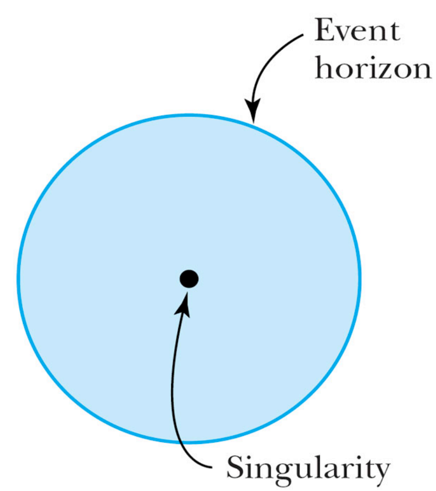
Note that, when ***R_s = 2GM/c²*** (called the **Schwarzschild radius**), this becomes

$$\Delta s^2 = \infty\,\Delta r^2 + r^2 \Delta\Omega^2 - 0\,\Delta t^2 \qquad (A.48)$$

When a star's thermonuclear fuel is depleted, no heat is left to counteract the force of gravity, which becomes dominant. The star's mass collapses into an incredibly dense ball that could warp space-time enough to not allow light to escape. The point at the center is called a ***singularity***.

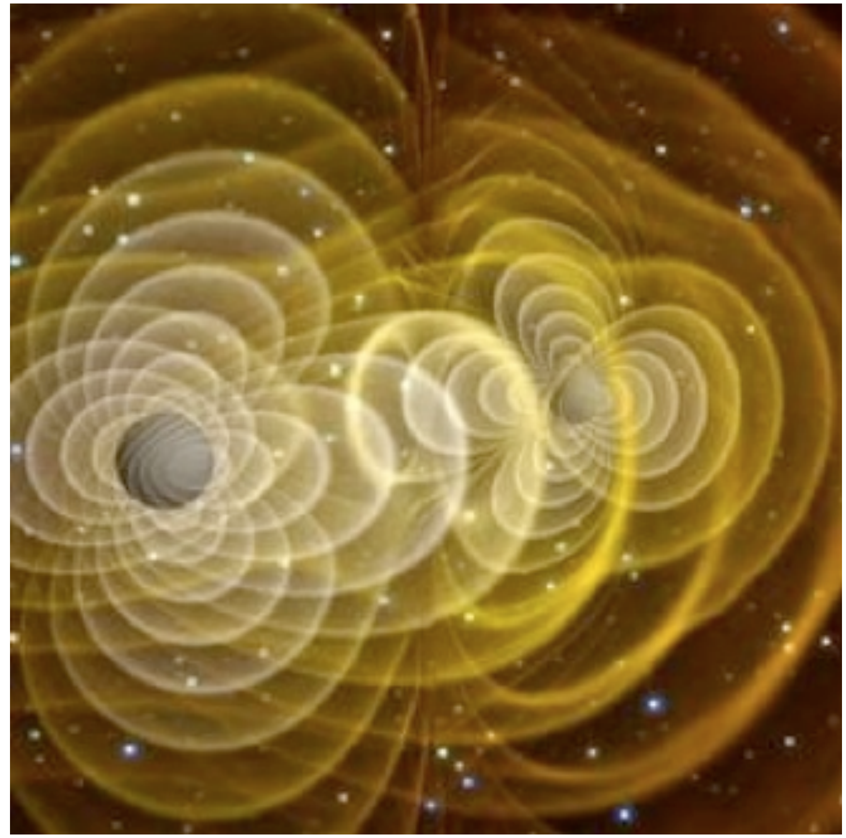
A collapsing star greater than 3 solar masses will distort space-time in this way to create a **black hole**.

Schwarzschild determined the radius of a black hole, known as the ***event horizon***. The Schwarzschild radius is given by Eq. (4.24) in Lecture 4.



Gravitational Waves

When a **charge accelerates**, the electric field surrounding the charge redistributes itself. This change in the electric field **produces an electromagnetic wave**, which is easily detected. Similarly, an **accelerated mass** should also **produce gravitational waves**.



Gravitational waves carry energy and momentum, travel at the speed of light, and are characterized by frequency and wavelength.

As gravitational waves pass through space-time, they cause small ripples. The stretching and shrinking is on the order of 1 part in 10^{21} even due to a strong gravitational wave source.

Due to their small magnitude, gravitational waves are difficult to detect. Large astronomical events could create measurable space-time waves such as the collapse of a neutron star, a black hole or the Big Bang.