Physics of Radioactive Beams[1]
Chapter 5
Nucleus-nucleus scattering at high energies

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0.1 Eikonal wavefunction

The free-particle wavefunction

$$\psi \sim e^{i\mathbf{k}\cdot\mathbf{r}} \tag{1}$$

becomes "distorted" in the presence of a potential $V(\mathbf{r})$. The distorted wave (DW) can be calculated numerically by solving the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\,\psi(\mathbf{r}) \tag{2}$$

with the condition that asymptotically $\psi(\mathbf{r})$ behaves as 1. This is done after a partial wave decomposition of $\psi(\mathbf{r})$, since in most situations $V(\mathbf{r}) \equiv V(r)$ is spherically symmetric. For each partial wave the equation to be solved is

$$\left[\frac{d^2}{dr^2} + k_\ell^2(r)\right]\chi_\ell(r) = 0\ , \tag{3}$$

where

$$k_\ell(r) = \left\{\frac{2m}{\hbar^2}\left[E - V(r) - \frac{\ell(\ell+1)\hbar^2}{2m\, r^2}\right]\right\}^{1/2} . \tag{4}$$

The asymptotic behavior of $\chi_\ell(r)$ is

$$\chi_\ell(r) \sim \sin\left(kr - \eta\ln 2kr - \frac{1}{2}\pi\ell + \delta_\ell\right) , \tag{5}$$

where δ_ℓ is the phase-shift and

$$k = \frac{m\mathrm{v}}{\hbar}, \qquad\qquad \eta = \frac{Z_1 Z_2 e^2}{\hbar\mathrm{v}}. \tag{6}$$

The phase-shifts are found by solving Eq. 3 numerically and by matching the solution with the asymptotic Eq. 5. Only for a pure Coulomb field is there an analytic solution, namely

$$\delta_\ell^C = \arg\Gamma(\ell + 1 + i\eta). \tag{7}$$

The distorted wavefunction is composed of an incident plane wave and a scattered wave

$$\psi \cong e^{i\mathbf{k}\cdot\mathbf{r}} + f(\theta)\frac{e^{ikr}}{r}\ , \tag{8}$$

where is $f(\theta)$ called the *scattering amplitude*. The scattering amplitude can be expanded in terms of partial waves:

$$f(\theta) = \frac{i}{2k}\sum_\ell (2\ell+1)\left[1 - e^{2i\,\delta_\ell}\right]P_\ell(\cos\theta). \tag{9}$$

It is also convenient to write the scattering amplitude as

$$f(\theta) = f_C(\theta) + f_N(\theta) \tag{10}$$

where

$$\begin{aligned} f_C(\theta) &= \frac{i}{2k}\sum_\ell (2\ell+1)\left[1 - e^{2i\,\delta_\ell^C}\right] P_\ell(\cos\theta) \\ &= -\frac{a_0}{2}\,\frac{1}{\sin^2\,\theta/2}\ \exp\left\{-i\eta\,\ln\left(\sin^2\,\frac{\theta}{2}\right) + 2i\delta_0^C\right\} \end{aligned} \tag{11}$$

is the Coulomb scattering amplitude, and

$$f_N(\theta) = \frac{i}{2k}\sum_\ell (2\ell+1)e^{2i\delta_\ell^c}\left[1 - e^{2i\,\delta_\ell^N}\right]\,P_\ell(\cos\theta) \tag{12}$$

is the nuclear scattering amplitude. In the equation above

$$\delta_\ell = \delta_\ell^C + \delta_\ell^N \tag{13}$$

and

$$a_0 = \frac{Z_1 Z_2\, e^2}{2E}\ . \tag{14}$$

In this formalism, the total reaction cross section is given by

$$\sigma_R = \frac{\pi}{k^2}\sum_\ell\,(2\ell+1)\left[1 - \exp\left(-2\mathrm{Im}\delta_\ell^N\right)\right]\ . \tag{15}$$

The quantity $\exp\left\{-2\mathrm{Im}\,\delta_\ell^N\right\}$ is often called the *transmission coefficient* and is an attenuation effect due to the imaginary part of the optical potential.

The solution of 3 (to obtain δ_ℓ) involves a great numerical effort, especially at large bombarding energies E. Fortunately, at large energies E a very useful approximation is valid when the excitation energies ΔE are much smaller than E and the nuclei (or nucleons) move in forward directions, i.e., $\theta \ll 1$.

Calling $\mathbf{r} = (z, \mathbf{b})$, where z is the coordinate along the beam direction, we can assume that

$$\psi(\mathbf{r}) = e^{ikz}\,\phi(z, \mathbf{b}), \tag{16}$$

where ϕ is a slowly varying function of z and b, so that

$$\left|\nabla^2\phi\right| \ll k\,|\nabla\phi|\,. \tag{17}$$

In cylindrical coordinates 2 becomes

$$2ik\, e^{ikz}\frac{\partial\phi}{\partial z} + e^{ikz}\frac{\partial^2\phi}{\partial z^2} + e^{ikz}\,\nabla_b^2\phi - \frac{2m}{\hbar^2}\,V\,e^{ikz}\,\phi = 0$$

or, neglecting the 2nd and 3rd terms because of 17,

$$\frac{\partial\phi}{\partial z} = -\frac{i}{\hbar \mathrm{v}}V(\mathbf{r}\,)\phi \tag{18}$$

whose solution is

$$\phi = \exp\left\{-\frac{i}{\hbar\mathrm{v}}\int_{-\infty}^{z} V(\mathbf{b}, z')dz'\right\}. \tag{19}$$

That is,

$$\psi(\mathbf{r}) = \exp\left\{ikz + i\chi(\mathbf{b}, z)\right\}, \tag{20}$$

where

$$\chi(\mathbf{b}, z) = -\frac{1}{\hbar\mathrm{v}}\int_{-\infty}^{z} V(\mathbf{b}, z')dz' \tag{21}$$

is the *eikonal phase*. Given $V(\mathbf{r}\,)$ one needs a single integral to determine the wavefunction: a great simplification of the problem.

The eikonal approximation, in the same form as given by Eqs. 20, can be obtained from the Klein-Gordon equation with a (scalar) potential V. The proof can be found in some textbooks, e.g., Ref. [2].

Supplement A

0.1.1 Green's function

The solution of the Schrödinger equation for $E > 0$ (*scattering states*) is denoted by $\psi_{\mathbf{k}}^{+}(r\,)$ where the plus sign indicates that the asymptotic behavior is that of an outgoing spherical wave. $\psi_{\mathbf{k}}^{-}(r\,)$ is also a solution of the same Schrödinger equation but has the asymptotic behavior of an incoming spherical wave. From time-reversal invariance

$$\psi_{\mathbf{k}}^{-*}(\mathbf{r}\,) = \psi_{-\mathbf{k}}^{+}(\mathbf{r}\,), \tag{22}$$

which gives, from Eq. 8,

$$\psi_{\mathbf{k}}^{-}(\mathbf{r}\,) \underset{r\to\infty}{\longrightarrow} e^{i\mathbf{k}\cdot\mathbf{r}} + f^*(\pi - \theta)\frac{e^{-ikr}}{r}. \tag{23}$$

Thus, we concentrate on $\psi_{\mathbf{k}}^{+}$ and if the sign is not given $\psi_{\mathbf{k}}^{+}$ is understood. We denote $\phi_{\mathbf{k}} = e^{i\mathbf{k}\cdot\mathbf{r}}$.

The Schrödinger equation is

$$(E - H_0)\psi_{\mathbf{k}}(\mathbf{r}) = V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}) \tag{24}$$

where H_0 is the kinetic energy operator. $\phi_{\mathbf{k}}$ obeys the equation

$$(E - H_0)\phi_{\mathbf{k}} = 0. \tag{25}$$

It also obeys the equation

$$(E - H_0)\phi_{\mathbf{k}'} = (E - E')\phi_{\mathbf{k}'}, \tag{26}$$

the orthogonality condition

$$\int \phi^*_{\mathbf{k}}(\mathbf{r}\,')\phi_{\mathbf{k}'}(\mathbf{r})d^3r = (2\pi)^3\delta(\mathbf{k} - \mathbf{k}'), \tag{27}$$

and the closure relation

$$\int \phi^*_{\mathbf{k}}(\mathbf{r}\,')\phi_{\mathbf{k}}(\mathbf{r})d^3k = (2\pi)^3\delta(\mathbf{r} - \mathbf{r}\,'). \tag{28}$$

It is useful to expand the scattering solution in terms of $\phi_{\mathbf{k}}$,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \int a(\mathbf{k}')\phi_{\mathbf{k}'}(\mathbf{r})d^3k'. \tag{29}$$

Using this in Eq. 26, Eq. 24 becomes

$$\int a(\mathbf{k}')(E - E')\phi_{\mathbf{k}'}(\mathbf{r})d^3k' = V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}). \tag{30}$$

Multiplying by $\phi^*_{\mathbf{k}}(\mathbf{r})$, integrating over $\mathbf{r}$, and using 28 we get

$$(2\pi)^3(E - E')a(\mathbf{k}') = \int \phi^*_{\mathbf{k}'}(\mathbf{r})V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r})d^3r. \tag{31}$$

Putting this back into Eq. 29 we have

$$\psi_{\mathbf{k}}(\mathbf{r}) = \int a(\mathbf{k}')\phi_{\mathbf{k}'}(\mathbf{r})d^3k' = \int G_0(\mathbf{r}, \mathbf{r}\,')V(\mathbf{r}\,')\psi_{\mathbf{k}}(\mathbf{r}\,')d^3r' \tag{32}$$

where $G_0(\mathbf{r}, \mathbf{r}\,')$ is given by

$$G_0(\mathbf{r}, \mathbf{r}\,') = \frac{1}{(2\pi)^3}\int \frac{\phi_{\mathbf{k}'}(\mathbf{r})\phi^*_{\mathbf{k}'}(\mathbf{r}\,')}{E - E'}d^3k'. \tag{33}$$

The general solution of Eq. 24 is obtained by adding any solution of the homogeneous Eq. 25 to Eq. 32. I.e.,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + \int G_0(\mathbf{r}, \mathbf{r}\,')V(\mathbf{r}\,')\psi_{\mathbf{k}}(\mathbf{r}\,')d^3r'. \tag{34}$$

Using 26 we obtain that $G_0(\mathbf{r},\mathbf{r}')$ satisfies the equation

$$(E-H_0)\, G_0(\mathbf{r},\mathbf{r}') = \frac{1}{(2\pi)^3}\int e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')} d^3k = \delta(\mathbf{r}-\mathbf{r}'). \tag{35}$$

Eq. 33 has poles in $k=\pm k'$. We can eliminate these poles by defining $G_0(\mathbf{r},\mathbf{r}')$ as

$$G_0^{\pm}(\mathbf{r},\mathbf{r}') = \lim_{\eta\to 0^{\pm}} \frac{1}{(2\pi)^3}\,\frac{2m}{\hbar^2}\int \frac{\phi_{\mathbf{k}'}(\mathbf{r})\phi^*_{\mathbf{k}'}(\mathbf{r}')}{k^2-k'^2\pm i\eta}\, d^3k' \tag{36}$$

where η is a small positive quantity which is allowed to go to zero after the integration is performed. The integral in 36 is easily obtained by the residue technique. One gets

$$G_0^{\pm}(\mathbf{r},\mathbf{r}') = -\frac{m}{2\pi\hbar^2}\,\frac{e^{\pm ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \tag{37}$$

The $\pm$ sign arises from the residues in the k'-plane at $+\eta$ and $-\eta$, respectively. Assuming that $V(\mathbf{r}')$ falls off rapidly with r' so that in the asymptotic region $r\gg r'$, we can approximate

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'|^{-1} \quad &\rightarrow \quad \frac{1}{r} \\ \exp\{\pm ik|\mathbf{r}-\mathbf{r}'|\} \quad &\rightarrow \quad \exp\{\pm i(kr-\mathbf{k}'\cdot\mathbf{r}')\} \end{aligned} \tag{38}$$

where $\mathbf{k}'=k\hat{\mathbf{r}}$.

Using 37, 38, and 34 we get

$$\psi_{\mathbf{k}}^{\pm}(\mathbf{r}) \underset{r\to\infty}{\longrightarrow} e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\,\frac{e^{\pm ikr}}{r}\int e^{\mp i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\psi_{\mathbf{k}}^{\pm}(\mathbf{r}')d^3r'. \tag{39}$$

Thus the $\pm$ sign in $G_0^{\pm}$ is associated with the $\pm$ sign of $\psi_{\mathbf{k}}$.

As a by-product we obtain for the scattering amplitude

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{-i\mathbf{k}'\cdot\mathbf{r}'} V(\mathbf{r}')\psi_{\mathbf{k}}^{+} d^3r'. \tag{40}$$

Multiplying 26 by $(E-H_0)^{-1}$ we get

$$\frac{1}{E-H_0}\,(E-H_0)\phi_{\mathbf{k}'}(\mathbf{r}) = (E-E')\,\frac{1}{E-H_0}\phi_{\mathbf{k}'}(\mathbf{r})$$

from which we obtain

$$\frac{1}{E-H_0}\phi_{\mathbf{k}'}(\mathbf{r}) = \frac{1}{E-E'}\phi_{\mathbf{k}'}(\mathbf{r}). \tag{41}$$

Thus, $\phi_{\mathbf{k}'}$ are eigenfunctions of the operator $(E - H_0)^{-1}$, with eigenvalues $1/(E - E')$ where $E' > 0$ and $E \neq E'$. For $E = E'$ the operator $(E - H_0)^{-1}$ is not defined. Similarly, from Eq. 24 we obtain

$$\psi_{\mathbf{k}}(\mathbf{r}) = (E - H_0)^{-1} V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}).$$

Removing the divergence by inserting a small quantity η and adding the solution of the homogeneous equation we get the formal solution

$$\psi_{\mathbf{k}}^{\pm}(\mathbf{r}) = \phi_{\mathbf{k}}(\mathbf{r}) + (E - H_0 \pm i\eta)^{-1} V(\mathbf{r})\psi_{\mathbf{k}}(\mathbf{r}). \tag{42}$$

This is a solution of the Schrödinger equation only when $\eta \to 0$.

Comparing 34 with 42 we see that formally the Green's function is given by

$$G_0^{\pm} = \frac{1}{E - H_0 \pm i\eta}. \tag{43}$$

We can check this by using Eq. 41 which can be written in the more general form

$$\frac{1}{E - H + i\eta}|m\rangle = \frac{1}{E - E_m + i\eta}|m\rangle$$

where H is any Hamiltonian and $|m\rangle$ are the corresponding eigenfunctions. Multiplying on the right by $\langle m|$, summing over m and using closure $(\sum_m |m\rangle\langle m| = 1)$ we get

$$\frac{1}{E - H + i\eta} = \sum_m \frac{1}{E - E_m + i\eta}|m\rangle\langle m| \tag{44}$$

where $\sum_m$ implies summation over discrete eigenstates and integration over continuous eigenstates. If H is replaced by the kinetic energy operator H_0 and $|m\rangle$ are replaced by the continuous eigenstates $(2\pi)^{-3/2}|\mathbf{k}\rangle$ equation 44 reduces to

$$\frac{1}{E - H_0 + i\eta} = \frac{1}{(2\pi)^3}\int d^3k' \, \frac{|\mathbf{k}'\rangle\langle\mathbf{k}'|}{E - E' + i\eta} \tag{45}$$

which is identical with Eq. 36, $G_0(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r}|G_0|\mathbf{r}'\rangle$. Eq. 42 can also be written formally as

$$|\psi^{\pm}\rangle = |\phi\rangle + G_0^{\pm} V |\psi^{\pm}\rangle. \tag{46}$$

Eq. 46 is know as the *Lipmann-Schwinger equation*. It can be solved by iteration, first replacing $|\psi^{\pm}\rangle$ by $|\phi\rangle$ on its r.h.s., then replacing by $|\phi\rangle + G_0^{\pm} V|\phi\rangle$, and so on. This is yields the *Lipmann-Schwinger series.*

0.2 Elastic scattering of spinless particles

The "outgoing" wavefunction of a particle scattered by a potential $V(r)$ is given by Eq. 34, i.e.,

$$\psi^+(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}\, V(\mathbf{r}')\psi^+(\mathbf{r}')d^3r', \tag{47}$$

where m is the particle mass.

For $r \gg r'$, we expand $|\mathbf{r}-\mathbf{r}'| \simeq r - \mathbf{r}\cdot\mathbf{r}'/r$ and obtain

$$\psi^+(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} - f(\theta)\frac{e^{+ikr}}{r} \tag{48}$$

where $f(\theta)$ is given by 40.

Using the eikonal wavefunction 20 for $\psi^+(\mathbf{r})$ in Eq. 40, we obtain

$$\begin{aligned} f(\theta) \quad &= \quad -\frac{m}{2\pi\hbar^2}\int d^2b\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{b}}\int_{-\infty}^{\infty} dz\, e^{i(\mathbf{k}-\mathbf{k}').\hat{\mathbf{k}}\, z} \\ &\times \; V(\mathbf{r})\exp\left\{-\frac{i}{\hbar \mathrm{v}}\int_{-\infty}^{z} V(\mathbf{r}')dz'\right\}. \end{aligned} \tag{49}$$

But, for $\theta \ll 1$, $(\mathbf{k}-\mathbf{k}')\cdot\mathbf{k} \cong 0$, and

$$\begin{aligned} &\int_{-\infty}^{\infty} dz\, V(\mathbf{r})\exp\left\{-\frac{i}{\hbar \mathrm{v}}\int_{-\infty}^{z} V(\mathbf{r}')dz'\right\} \\ &= \; i\hbar \mathrm{v}\exp\left\{-\frac{i}{\hbar \mathrm{v}}\int_{-\infty}^{z} V(\mathbf{r}')dz'\right\}\Bigg|_{-\infty}^{\infty} = i\hbar \mathrm{v}\left\{e^{i\chi(\mathbf{b})}-1\right\} \end{aligned}$$

where

$$\chi(\mathbf{b}) \equiv \chi(\mathbf{b},\infty) = -\frac{1}{\hbar \mathrm{v}}\int_{-\infty}^{\infty} V(\mathbf{b},z')\, dz' \tag{50}$$

is the total eikonal phase.

Denoting

$$\mathbf{q} \equiv \Delta\mathbf{k} = \mathbf{k}-\mathbf{k}', \qquad q = 2k\sin\theta/2 \tag{51}$$

we get

$$f(\theta) = -\frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\left[e^{i\chi(\mathbf{b})}-1\right]. \tag{52}$$

If the potential is spherically symmetric, $\chi(b)$ is a function of the absolute value of b only and using

$$\int_0^{2\pi} d\phi\, e^{iqb\cos\phi} = 2\pi\, J_0(qb)\,,$$

we obtain,

$$f(\theta) = -ik \int db\ b\ J_0(qb) \left[e^{i\chi(b)} - 1\right] . \tag{53}$$

Thus in the eikonal approximation the elastic scattering amplitude $f(\theta)$ is obtained from two simple integrals: Eqs. 50 and 53.

The *elastic scattering* cross section is given by

$$\frac{d\sigma_{el}}{d\theta} = |f(\theta)|^2 \tag{54}$$

Supplement B

0.2.1 Coulomb amplitude and Coulomb eikonal phase

In general, the scattering potential is given by

$$V(\mathbf{r}) = U_N^{\rm opt}(r) + U_C(r) \tag{55}$$

where $U_N^{\rm opt}$ is the nuclear optical potential and $U_C(r) = Z_1 Z_2 e^2/r$ is the Coulomb potential between the nuclei.

Since $U_N^{\rm opt}$ (generally complex) is well localized in space, the eikonal phase for the nuclear part of 55 is obtained by a well convergent integral Eq. 50. However, the integral diverges logarithmically for the Coulomb potential. This is due to the use of the approximation 17 which is not valid for the (long range) Coulomb potential. But this does not pose a real problem since an analytical formula can be given for the Coulomb eikonal phase which reproduces the exact Coulomb amplitude, Eq. 11. The total eikonal phase is

$$\chi(b) = -\frac{1}{\hbar \mathrm{v}} \int_{-\infty}^{\infty} U_N^{\rm opt}(b, z')dz' + \chi_C(b) \tag{56}$$

where

$$\chi_C(b) = \frac{2Z_1 Z_2 e^2}{\hbar \mathrm{v}}\ \ln(kb) \tag{57}$$

We now show that the Coulomb phase, as given by the above formula, reproduces the Coulomb amplitude in the eikonal approximation. We have

$$\begin{aligned} f_C(\theta) &= ik\int_0^\infty J_0(qb)\left[e^{i\chi_C(b)}-1\right]bdb \\ &= \frac{ik}{q^2}\int_0^\infty J_0(x)\left[\left(\frac{kx}{q}\right)^{i2\eta}-1\right]xdx \\ &= \frac{ik}{q^2}\left\{\left(\frac{k}{q}\right)^{i2\eta}\int_0^\infty J_0(x)x^{i2\eta+1}dx-\int_0^\infty J_0(x)xdx\right\} \end{aligned}$$

where

$$\eta=\frac{Z_1Z_2e^2}{\hbar \mathrm{v}};\qquad k=\frac{m\mathrm{v}}{\hbar};\qquad q=2k\sin(\theta/2)\,. \tag{58}$$

Integrating by parts and using

$$\int xJ_0(x)dx=x\,J_1(x)$$

one gets

$$\begin{aligned} f(\theta) &= \frac{ik}{q^2}\left(\frac{k}{q}\right)^{i2\eta}x^{i2\eta+1}J_1(x)\Big|_0^\infty-i2\eta\left(\frac{k}{q}\right)^{i2\eta}\int_0^\infty x^{i2\eta}J_1(x)dx-xJ_1(x)\Big|_0^\infty \\ &= \frac{ik}{q^2}\left\{xJ_1(x)\,\left[e^{i2\eta\ln(kx/q)}-1\right]\Big|_\infty^0-2i\eta\left(\frac{k}{q}\right)^{i2\eta}\cdot 2^{i2\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}\right\} \end{aligned}$$

where we have used the integral Eq. 6.561.14 of the book of Gradshteyn and Rhyzik [3] in the last step.

The first term in the equation above can be neglected and we get

$$f_C(\theta)=\frac{2k\eta}{q^2}\left(\frac{2k}{q}\right)^{i2\eta}\frac{\Gamma(1+i\eta)}{\Gamma(1-i\eta)}$$

But

$$\Gamma(1\pm i\eta)=\pm i\eta\Gamma(\pm i\eta)=\pm i\eta\left|\Gamma(i\eta)\right|\left\{\begin{array}{l} e^{i\phi_0} \\ e^{-i(\phi_0+\pi)}\end{array}\right.$$

where

$$\phi_0=\arg\Gamma(1+i\eta) \tag{59}$$

Therefore,

$$f_C(\theta)=-\frac{Z_1Z_2e^2}{2m\mathbf{v}^2\sin^2\frac{\theta}{2}}\,\exp\left\{-i\eta\ln\left(\sin^2\frac{\theta}{2}\right)+i2\phi_0\right\} \tag{60}$$

which is the exact Coulomb amplitude, Eq. 11 (with $\phi_0\equiv\delta_0^C$).

The phase 59 can also be written as

$$\phi_0 = -\eta\gamma + \sum_{k=0}^{\infty} \left(\frac{\eta}{k+1} - \arctan \frac{\eta}{k+1} \right) \tag{61}$$

where $\gamma = 0.57721\ldots$ is the Euler's constant.

For numerical evaluation it is appropriate to rewrite Eq. 53 as

$$f(\theta) = ik \int_0^\infty J_0(qb)\, e^{i\chi_C(b)} \left[1 - e^{i\chi_N(b)}\right] bdb + f_C(\theta) \tag{62}$$

which can be easily obtained by adding and subtracting f_C to 53 and combining terms. This is because $1 - \exp\left[i\chi_N(b)\right]$ drops to zero rapidly for $b > R_1 + R_2$ since $U_N^{\rm opt}$ goes to zero there. In Eq. 62 we use 60 for $f_C(\theta)$. $\chi_N(b)$ is given by the first term of Eq. 56 and $\chi^C(b)$ is given by 57.

A few modifications of the Coulomb eikonal phase are need to account for the extended nature of the nuclear charge distributions. For light nuclei, one can assume Gaussian nuclear densities, and the Coulomb phase is given by

$$\chi_C(b) = 2\ \frac{Z_1 Z_2 e^2}{\hbar v}\ \left[\ln(kb) + \frac{1}{2} E_1 \left(\frac{b^2}{R_G^2} \right) \right], \tag{63}$$

with $R_G^{(i)}$ equal to the size parameter of Gaussian matter densities of nucleus 1 and nucleus 2, respectively, $R_G^2 = [R_G^{(1)}]^2 + [R_G^{(2)}]^2$, and

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t}\ dt\ . \tag{64}$$

The first term in Eq. (63) is the contribution to the Coulomb phase of a point-like charge distribution. It reproduces the elastic Coulomb amplitude when introduced into the eikonal expression for the elastic scattering amplitude, as we have shown above. The second term in Eq. (63) is a correction due to the extended Gaussian charge distribution. It eliminates the divergence of the Coulomb phase at $b = 0$, so that

$$\psi_C(0) = 2\frac{Z_1 Z_2 e^2}{\hbar v}\ \left[\ln(kR_G) - \gamma\right] \tag{65}$$

where γ is the Euler's constant.

For heavy nuclei a "black-sphere" absorption model is more appropriate. Assuming an absorption radius R_0, the Coulomb phase is given by

$$\begin{aligned}\chi_C(b) = 2\frac{Z_1 Z_2 e^2}{\hbar v}\ \Big\{ &\Theta(b - R_0) \ln(kb) + \Theta(R_0 - b)\ \Big[\ln(kR_0) \\ &+ \ln\left[1 + (1 - b^2/R_0^2)^{1/2}\right] - (1 - b^2/R_0^2)^{1/2} - \frac{1}{3}\ (1 - b^2/R_0^2)^{3/2} \Big] \Big\}\ . \end{aligned} \tag{66}$$

Again, the first term inside the curly brackets is the Coulomb eikonal phase for pointlike charge distributions. The second term accounts for the finite extension of the charge distributions.

0.3 Spin particles

In the scattering of a spin-$\frac{1}{2}$ particle by a nucleus, e.g., proton-nucleus scattering, the optical potential contains a spin-orbit term, i.e.,

$$U_N^{\text{opt}} = U_N(r) + U_S(r)\sigma\cdot\mathbf{L} \tag{67}$$

where

$$\mathbf{L} = \frac{1}{\hbar}\left(\mathbf{r}\times\mathbf{p}\right), \qquad\qquad \mathbf{s} = \frac{1}{2}\sigma, \tag{68}$$

and the Dirac matrices $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ obey the commutation rule

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}. \tag{69}$$

As in the spinless case, the distorted wavefunction can be written as

$$\psi_{\mathbf{r}}(\mathbf{r}) \cong e^{i\mathbf{k}\cdot\mathbf{r}}\, \varphi(\mathbf{r})u_i(\mathbf{k}) \tag{70}$$

where $u_i(\mathbf{k})$ is a spinor wavefunction. One obtains

$$\varphi(\mathbf{r}) \cong \exp\left\{-\frac{i}{\hbar \mathrm{v}}\int_{-\infty}^{z} dz'\left\{U_N(b,z') + U_S(b,z')\sigma\cdot(\mathbf{b}\times\hat{\mathbf{k}})k\right\}\right\} \tag{71}$$

since $\mathbf{z} = z\hat{\mathbf{k}}$.

The eikonal amplitude is

$$f(\theta) = \frac{k}{2\pi i}\int e^{i\mathbf{q}\cdot\mathbf{b}}\left\{e^{i\chi_N(b)+i\chi_S(b)\sigma\cdot(\mathbf{b}\times\hat{\mathbf{k}})k} - 1\right\} d^2b \tag{72}$$

where

$$\chi_N(b) = -\frac{1}{\hbar \mathrm{v}}\int_{-\infty}^{\infty} U_N(b,z)dz, \qquad \chi_S(b) = -\frac{1}{\hbar \mathrm{v}}\int_{-\infty}^{\infty} U_S(b,z)dz\ . \tag{73}$$

Using 69

$$\begin{aligned}
\exp\left\{i\chi_S(b)\sigma\cdot(\mathbf{b}\times\hat{\mathbf{k}})k\right\} &= \sum_n \frac{1}{n!}\left[i\chi_S(b)\sigma\cdot(\mathbf{b}\times\hat{\mathbf{k}})k\right]^n \\
= \quad \sum_{n=\text{odd}} \frac{1}{n!}(kb)^n\left[i\chi_S(b)\right]^n \sigma\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{k}}) &+ \sum_{n=\text{even}}\frac{1}{n!}(kb)^n\left[i\chi_S(b)\right]^n \\
= \quad i\sigma\cdot(\hat{\mathbf{b}}\times\hat{\mathbf{k}})\sin\left[kb\chi_S(b)\right] &+ \cos\left[kb\chi_S(b)\right].
\end{aligned} \tag{74}$$

Including the Coulomb amplitude as in 62 we obtain

$$f(\theta) = \frac{ik}{2\pi}\int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}}\, e^{i\chi_C(b)}\left[1 - e^{i\chi_N(b)+i\chi_S(b)\,\sigma\cdot(\mathbf{b}\times\mathbf{k}}\,)\right] + f_C(\theta) \tag{75}$$

which, by using 74 becomes

$$f(\theta) = \frac{ik}{2\pi} \int d^2b\, e^{i\mathbf{q}\cdot\mathbf{b}} \left\{ \Gamma_0(b) + i\sigma \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{k}})\Gamma_1(b) \right\} + f_C(\theta) \tag{76}$$

where the *profile functions* $\Gamma_0(b)$ and $\Gamma_1(b)$ are defined as

$$\begin{aligned} \Gamma_0(b) &= e^{i\chi_C(b)} \left\{ 1 - e^{i\chi_N(b)} \cos\left[kb\chi_S(b)\right] \right\}, \\ \Gamma_1(b) &= -e^{i\chi_C(b) + i\chi_N(b)} \sin\left[kb\chi_S(b)\right]. \end{aligned} \tag{77}$$

It is more convenient to rewrite 76 as

$$f(\theta) = F(\theta) + (\sigma \cdot \hat{\mathbf{n}})G(\theta) \tag{78}$$

where $\hat{\mathbf{n}} = \dfrac{\mathbf{k} \times \mathbf{k}'}{|\mathbf{k} \times \mathbf{k}'|}$. The azimuthal integrals in 76 can be easily performed,[1]

$$\begin{aligned} F(\theta) &= f_C(\theta) + ik \int_0^\infty J_0(qb)\Gamma_0(b) b db \\ G(\theta) &= -ik \int_0^\infty J_1(qb)\Gamma_1(b) b db. \end{aligned} \tag{79}$$

Since $J_1(x=0) = 0, \quad G(\theta = 0) = 0$, which is a consequence of the conservation of angular momentum + spin.

For unpolarized beams

$$\frac{d\sigma_{\rm el}}{d\Omega} = \frac{1}{2} \sum_{\rm spins} |f(\theta)|^2 = |F(\theta)|^2 + |G(\theta)|^2. \tag{80}$$

0.4 Total reaction cross sections

According to the unitary theorem (see [4]) the total scattering cross section is given by

$$\sigma_{\rm tot} = \frac{4\pi}{k} \mathrm{Im} f(\theta = 0^{\rm o}). \tag{81}$$

Using 52 we obtain

$$\sigma_{\rm tot} = 2 \int \left[1 - \mathrm{Re}\, e^{i\chi(\mathbf{b})} \right] d^2b. \tag{82}$$

[1] $\sigma \cdot (\mathbf{b} \times \hat{\mathbf{k}}) = (\sigma \cdot \hat{\mathbf{n}}) b_{||}$, where $b_{||}$ is the component of $\mathbf{b}$ parallel to $\mathbf{k}'$. So, $\sigma \cdot (\mathbf{b} \times \hat{\mathbf{k}}) = (\sigma \cdot \hat{\mathbf{n}}) b \cos\theta$. The component of $\mathbf{b}$ perpendicular to $\mathbf{k} \times \mathbf{k}'$ defines another term, proportional to $(\sigma \cdot \xi)$, which is unnecessary by symmetry arguments.

We can also verify 82 by an integration of 54 over angles, using the eikonal amplitude $f(\theta)$, 52,

$$\begin{aligned} \int |f(\theta)|^2 d\Omega_k &= \left(\frac{k}{2\pi}\right)^2 \int e^{i(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{b}-\mathbf{b}')} \left\{e^{i\chi(\mathbf{b})} - 1\right\} \\ &\quad \times \left\{e^{-i\chi(\mathbf{b}')} - 1\right\} d^2b\, d^2b'\, d\Omega_{k'}. \end{aligned} \tag{83}$$

But

$$\begin{aligned} d\Omega_{k'} &= \sin\theta d\theta d\phi = \frac{k^2 \sin\theta d\theta d\phi}{k^2} \cong \frac{(k\theta d\theta)\,(kd\phi)}{k^2} \qquad \text{valid for} \quad \theta \ll 1 \\ &\cong \frac{kd\phi dk}{k^2} = \frac{d^2k'}{k^2}. \end{aligned} \tag{84}$$

Furthermore, using

$$\int e^{(\mathbf{k}-\mathbf{k}')\cdot(\mathbf{b}-\mathbf{b}')} d^2k' = (2\pi)^2\, \delta^{(2)}(\mathbf{b}-\mathbf{b}') \tag{85}$$

where $\delta^{(2)}(\mathbf{b}-\mathbf{b}')$ is a two-dimensional delta-function, we find

$$\sigma_{\text{scatt}} = \int \left|e^{i\chi(\mathbf{b})} - 1\right|^2 d^2b. \tag{86}$$

For $\chi(\mathbf{b})$ real (real optical potential), 86 reduces to 82.

However, when $\chi(\mathbf{b})$ is complex (optical potential has an imaginary part). 86 is not equal to 82. The difference is equal to the reaction cross section. That is,

$$\sigma_R = \sigma_{\text{tot}} - \sigma_{\text{scatt}} = \int \left[1 - \left|e^{i\chi(\mathbf{b})}\right|^2\right] d^2b = \int \left[1 - T\,(\mathbf{b})\right] d^2b \tag{87}$$

where $T\,(\mathbf{b}) = \left|e^{i\chi(\mathbf{b})}\right|^2 = e^{-2\mathrm{Im}\chi(\mathbf{b})}$ is known as the *transparency function*.

Supplement C

0.5 The "$t\rho\rho$" approximation

The theory of multiple scattering has been studied by many other authors, besides Glauber. Particularly useful for our purposes are the results obtained by Kerman, McManus and Thaler [5].

For a review of the theory we refer to Ref. [6]. The basic idea is that the nucleus-nucleus potential at a separation distance R is proportional to a folding of the nucleon densities. The proportionality constant is the t-matrix for nucleon-nucleon scattering at forward angles, namely $t_{NN}(\theta = 0^o, E) = -\left(2\pi\hbar^2/\mu\right)\, f_{NN}(\theta = 0^o, E)$,where $\mu = m/2$. In other words

$$\begin{aligned} U_N^{\rm opt}(\mathbf{R}\,) &\cong t_{NN}(\theta = 0^o, E) \int d^3r \rho_1(\mathbf{r}\,)\rho_2(\mathbf{R} + \mathbf{r}\,) \\ &= -\frac{2\pi\hbar^2}{\mu} f_{NN}(\theta = 0^o) \int d^3r \rho_1(\mathbf{r}\,)\rho_2(\mathbf{R} + \mathbf{r}\,). \end{aligned} \tag{88}$$

This approximation assumes that only binary collisions between the nucleons occur and that each nucleon interacts once only. Multiple nucleon-nucleon collisions lead to corrections to 88 and involve nucleon-nucleon correlation distances in matter (see [6]).

From the optical theorem, $\mathrm{Im} f(\theta = 0^o) = (k/4\pi)\, \sigma_{NN}$, and defining

$$\alpha = \frac{\mathrm{Re} f_{NN}(\theta = 0^o)}{\mathrm{Im} f_{NN}(\theta = 0^o)} \tag{89}$$

we can rewrite 88 as

$$U_N^{\rm opt}(\mathbf{R}\,) = -\sigma_{NN}\, \frac{\hbar v}{2}\, (\alpha + i) \int d^3r\ \rho_1(\mathbf{r})\rho_2(\mathbf{R} + \mathbf{r}\,). \tag{90}$$

The value of the nucleon-nucleon cross section, σ_{NN} , which enters Eq. 90 is somewhat modified by the Pauli-blocking (for a study of the effect see Ref. [6]). But, since the elastic and inelastic cross sections are mainly dependent on the peripheral collisions between the nuclei, with poor nuclear overlap, σ_{NN} is not appreciably modified there, at least for highly energetic collisions. Thus, to construct the "$t\rho\rho$" optical potential only needs the values of σ_{NN}, α , and the ground state densities $\rho_i(r\,)$.

Good parametrization of the ground state nuclear densities are given by Gaussian distributions for light nuclei (e.g., $\alpha's$ and ^{12}C) or by Fermi functions for heavy nuclei (e.g. ^{40}C and ^{208}Pb).

Nucleus	Model	R (fm)	a (fm)	c (fm)
^{4}He	Gaussian	1.37		
^{12}C	MF	2.335	0.522	−0.149
^{16}O	MF	2.608	0.513	−0.051
^{20}Ne	MF	2.740	0.569	0
^{28}Si	MF	3.300	0.545	− 0.18
^{40}Ca	MF	3.725	0.591	− 0.169
^{42}Ca	MF	3.627	0.594	−0.102
^{58}Ni	MF	4.309	0.517	−0.131
^{90}Zr	MG	4.522	2.522	0.245
^{208}Pb	MF	6.624	0.549	0

Table 2.1 - Data for parameters used in Gaussian, Modified Fermi (MF), and Modified Gaussian (MG) fits of the density matter distribution of some nuclei (from Refs. [7] and [8]).

For a better description slight modifications of the Gaussian or Fermi distributions might be needed. In Table 2.1. we give examples of ground state nuclear densities for some nuclei, where we use

$$\begin{aligned} \rho(r) &= \rho(0)\, e^{-r^2/R^2} \quad , \qquad \text{(Gaussian)} \\ &= \rho(0)\left(1+\frac{cr^2}{R^2}\right)\left\{1+\exp\left[\left(r-R\right)/a\right]\right\}^{-1} \qquad \text{(modified Fermi (MF))} \\ &= \rho(0)\left(1+\frac{cr^2}{R^2}\right)\left\{1+\exp\left[\left(r^2-R^2\right)/a^2\right]\right\}^{-1} \qquad \text{(modified Gaussian (MG)).} \end{aligned} \tag{91}$$

For σ_{NN} and α_{NN}, the values in Table 2.2 were compiled [9, 10].

E/A	σ_{NN} (fm^2)	. α_{NN} .
30	19.6	0.87
38	14.6	0.89
40	13.5	0.9
49	10.4	0.94
85	6.1	1
94	5.5	1.07
120	4.5	0.7
200	3.2	0.6
342.5	2.84	0.26
550	3.6	0.04
1000	4.22	-0.2
2200	4.35	-0.3

Table 2.2 - Nucleon-nucleon cross sections. E/A is the laboratory energy per nucleon.

From Table 2.2 we observe that at very high energies ($\gtrsim$ 500 MeV/nucleon) the real part of the "$t\rho\rho$" potential vanishes, i.e.,

$$U_N^{\rm opt}(\mathbf{R}) = -i\sigma_{NN}\frac{\hbar\nu}{2}\int \rho_1(\mathbf{r})\rho_2(\mathbf{R}+\mathbf{r})d^3r \qquad [\gtrsim 500 \text{ MeV}]\,. \tag{92}$$

But at $E_{\rm lab}$/nucleon $\gtrsim$ 1 GeV the real part of the potential becomes repulsive (see Table 2.2). In any case, we can write

$$\sigma_R = \int [1-T(\mathbf{b})]d^2b \tag{93}$$

where

$$T(\mathbf{b}) = \exp\left[-\sigma_{NN}\int_{-\infty}^{\infty} dz \int \rho_1(\mathbf{r})\rho_2(\mathbf{R}+\mathbf{r})d^3r\right]. \tag{94}$$

Eq. 94 has a simple interpretation. The mean free path for a nucleon-nucleon collision is given by $\lambda_{NN}(R) = \left(\sigma_{NN}\int \rho_1(\mathbf{r})\,\rho_2(\mathbf{R}+\mathbf{r})d^3r\right)^{-1}$. Thus, the probability that the nuclei "survive" without a nucleon-nucleon collision is given by $\exp\left[-\int \dfrac{dz}{\lambda_{NN}(\mathbf{R})}\right]$ which is a product of the probabilities that the nuclei survive after moving through each path element dz along the trajectory (assumed to be a straight-line). This probabilistic interpretation is consistent with the concepts introduced in Chapter 1.

Thus $T(\mathbf{b})$ is known as the transparency function for a collision with impact parameter b. Of course, $1-T(\mathbf{b})$ is the probability that the nuclei interact at that impact parameter. Eq. 93 is then the result of two-body collisions in nucleus-nucleus collisions.

0.6 Inelastic scattering

The eikonal approximation is also very useful to calculate the excitation of a nucleus in a grazing collision with another one. When the excitation amplitude is small so that the Distorted wave Born approximation can be used (see Chapter 8), we can write

$$T_{if} = \left\langle \psi_{\mathbf{k}'}^{(-)}(\mathbf{r})\phi_f(\mathbf{r}')|V(\mathbf{r},\mathbf{r}')|\psi_{\mathbf{k}}^{(+)}(\mathbf{r})\phi_i(\mathbf{r}')\right\rangle \tag{95}$$

where $\psi_{\mathbf{k}',\mathbf{k}}^{(\pm)}$ are the (outgoing/ingoing) distorted scattered waves for the c.m. motion of the two nuclei and $\phi_{i,f}(\mathbf{r}')$ are the initial and final wavefunctions for the internal nuclear motion, respectively. We will now describe a method appropriate for inelastic scattering in high energy collisions, described in Ref. [11]. In the eikonal approximation

$$\psi_{\mathbf{k}'}^{(-)*}(\mathbf{r})\psi_{\mathbf{k}}^{(+)}(\mathbf{r}) = \exp\{-i\mathbf{q}\cdot\mathbf{r}+i\chi(\mathbf{b})\} \tag{96}$$

where $\mathbf{q}=\mathbf{k}'-\mathbf{k}$, $b=|\mathbf{r}\times\hat{\mathbf{k}}|$.

In 95 $V(\mathbf{r},\mathbf{r}')$ is the interaction potential between the two nuclei. It can be taken as the interaction potential between a nucleon in nucleus 1 at position r_1, with a nucleon in nucleus 2 at position r_2 (see Fig. 1). Thus,

$$T_{if} = \int d^3R\left\langle \phi_f^{(1)}(\mathbf{r}_1)\phi_f^{(2)}(\mathbf{r}_2)\left|V(\mathbf{r})\,e^{-i\mathbf{q}\cdot\mathbf{R}+i\chi(\mathbf{b})}\right|\phi_i^{(1)}(\mathbf{r}_1)\phi_i^{(2)}(\mathbf{r}_2)\right\rangle \tag{97}$$

where

$$\mathbf{r} = \mathbf{R}+\mathbf{r}_2-\mathbf{r}_1. \tag{98}$$

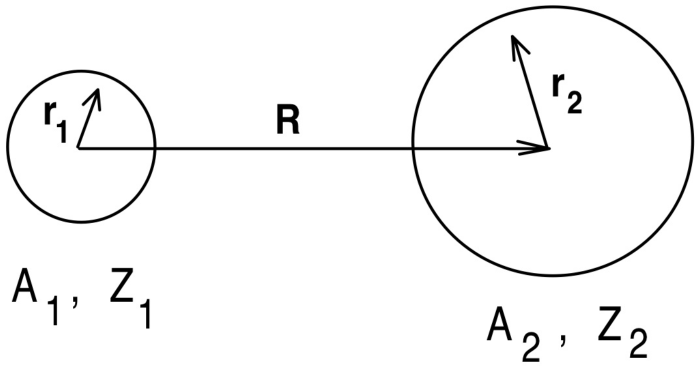

Figure 1: Coordinates used in text.

Using the Fourier transform

$$V(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3q \, V(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{r}} \tag{99}$$

the expression 95 becomes

$$T_{if} = \frac{1}{(2\pi)^3} \int d^3p \, d^3R \, e^{-i\mathbf{q}\cdot\mathbf{R}+i\mathbf{p}\cdot\mathbf{R}+i\chi(\mathbf{b})} \, \mathcal{M}(\mathbf{q}) \tag{100}$$

where

$$\mathcal{M}(\mathbf{q}) = \left\langle \phi_F^{(1)}(\mathbf{r}_1)\phi_f^{(2)}(\mathbf{r}_2) \left| e^{-i\mathbf{p}\cdot\mathbf{r}_1} V(\mathbf{q}) \, e^{i\mathbf{p}\cdot\mathbf{r}_2} \right| \phi_i^{(1)}(\mathbf{r}_1)\phi_i^{(2)}(\mathbf{r}_2) \right\rangle . \tag{101}$$

The z-integral can be done immediately, resulting in

$$T_{if} = \frac{1}{(2\pi)^2} \int d^2b \; e^{-i\mathbf{q}\cdot\mathbf{b}+i\chi(\mathbf{b})} \int d^2p_t \, e^{i\mathbf{p}_t\cdot\mathbf{b}} \mathcal{M}(\mathbf{p}) \tag{102}$$

where $\mathbf{p}$ is now given by $\mathbf{p} = \mathbf{p}_t + q_z\widehat{\mathbf{z}}$. The indices t and z refer to the direction perpendicular and parallel to the collision axis, respectively.

For a spherically symmetric optical potential, $\chi(\mathbf{b}) \equiv \chi(b)$ and the azimuthal integrals in Eq. 95 can be easily performed, resulting in

$$T_{if} = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} \int_0^\infty db \; b \; J_\nu(q_t b) e^{i\chi(b)} \int_0^\infty dp_t \; p_t \; J_\nu(q_t b) \int_0^{2\pi} d\phi_p \, e^{-i\nu\phi_p} \mathcal{M}(\mathbf{p}). \tag{103}$$

For small energy transfers, and in inter intermediate or high-energy collisions, the momentum transfer $\mathbf{p}$ is predominantly transverse. Thus,

$$T_{if} = \frac{1}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\phi} \int_0^\infty db\; b\; J_\nu(q_t b) M(\nu, b)\, e^{i\chi(b)} \tag{104}$$

where

$$M(\nu, b) = \int_0^\infty dp_t\; p_t J_\nu(p_t b) \int_0^{2\pi} d\phi_p\, e^{-i\nu\phi_p} \mathcal{M}(\mathbf{p}_t). \tag{105a}$$

The differential cross section for inelastic scattering is obtained by an average of initial spins and sum over final spins, i.e.,

$$\frac{d\sigma}{d\Omega} = \frac{k'}{k} \left(\frac{\mu}{2\pi\hbar^2}\right)^2 \frac{1}{(2j_1+1)(2j_2+1)} \sum_{\text{spins}} |T_{if}|^2 \tag{106}$$

where μ is the reduced mass of projectile + target. Performing the azimuthal integration,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{k'}{k} \left(\frac{\mu}{4\pi^2\hbar^2}\right)^2 (2j_1+1)^{-1} (2j_2+1)^{-1} \\ &\times \sum_\nu \sum_{\text{spins}} \left| \int_0^\infty db\; b\; J_\nu(q_t b) M(\nu, b)\, e^{i\chi(b)} \right|^2 . \end{aligned} \tag{107}$$

In high-energy collisions (see Eq. 84) $d\Omega \cong 2\pi\, q_t\, dq_t/k^2$. Using

$$\int J_\nu(q_t b)\; J_\nu(q_t b') q_t dq_t = \frac{1}{b} \delta(b - b') \tag{108}$$

we can write the total cross section as

$$\sigma = 2\pi \int_0^\infty db\; b\; \mathcal{P}_{if}(b) \tag{109}$$

where P_{if} is interpreted as the probability for inelastic excitation in a collision at impact parameter b:

$$\begin{aligned} \mathcal{P}_{if}(b) &= \frac{k'}{k} \left(\frac{1}{4\pi^2\hbar v}\right)^2 (2j_1+1)^{-1} (2j_2+1)^{-1} \\ &\times \exp\{-2\mathrm{Im}\chi(b)\} \sum_\nu \sum_{\text{spins}} |M(\nu, b)|^2 . \end{aligned} \tag{110}$$

The differential and total inelastic cross sections probe the structure of the nuclei through the matrix elements 101 and 105a. For some simple cases (see e.g. Ref. [11]) these matrix elements can be easily calculated.

0.7 Glauber theory of multiple scattering

The conditions of validity of the eikonal approximation are that the momentum and energy transfers in high energy collisions are much smaller than the bombarding energy. However, the eikonal-Born approximation described in last Section assumes that the transition $|i> \rightarrow |f>$ occurs in one step.

Glauber [1] has shown how to treat the general problem of multi-step collisions using the eikonal approximation. The derivation is quite similar to the one leading to Eq. 53 and we only present the result here. The inelastic amplitudes (note that $f^{\rm inel} = -\frac{\mu}{2\pi\hbar^2} T_{if}$)

$$f^{\rm inel}(\theta) = \frac{k}{2\pi i} \int d^2b \; e^{i\mathbf{q}\cdot\mathbf{b}} \int d^3r' \psi_f^*(\mathbf{r}') \; \left[e^{i\chi(\mathbf{b}-\mathbf{s})} - 1\right] \psi_i(\mathbf{r}') \tag{111}$$

where ψ_i (ψ_f) denote the initial (final) internal wavefunctions of the projectile, or target (or both) and

$$\mathbf{s} = \mathbf{r}' - \hat{\mathbf{k}}\,(\hat{\mathbf{k}} \cdot \mathbf{r}') \equiv \rho' \tag{112}$$

is the component of $\mathbf{r}'$ perpendicular to the propagation direction $\hat{\mathbf{k}}$.

One can easily show that 95 is a limit of 111 when $\chi \ll 1$, if the potential V in 95 is assumed to be the same as the one entering in the calculation of χ. However, in some situations one can make a clear distinction between the potential which induces the excitation and the one which leads to elastic scattering. An example of this is π-exchange in peripheral nuclear collisions [11]. In this case $V = V_{\pi-{\rm exch}}$ in 95, while the nuclei are scattered by a $U_n^{\rm opt} + U_c$. Then 95 is valid even for $\chi \gg 1$.

The advantage of using the Glauber amplitude 111 is to treat multiple collisions between the constituents of the nuclei. Then the amplitude is calculated from a single fundamental interaction, namely the nucleon-nucleon potential. Fortunately, as we shall see later, this interaction is not needed and in most situations the only inputs needed are the nucleon-nucleon cross sections and nuclear ground-state densities, which are well known.

For an explicitly many-particle system we replace the single particle wavefunction $\psi(\mathbf{r})$ in 111 by the many-particle wavefunction

$$\psi(\mathbf{r}) \rightarrow \psi(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n)$$

and the single-particle phase shift function $\chi(\mathbf{b})$ by the phase shift in the projectile wavefunction due to multiple collisions with the target nucleons. That is,

$$\chi(\mathbf{b} - \mathbf{s}) \rightarrow \sum_{j=1}^{n} \chi_j(\mathbf{b} - \mathbf{s}_j)$$

Thus, 111 becomes

$$f^{\text{inel}}(\theta) = \frac{k}{2\pi i}\int e^{i\mathbf{q}\cdot\mathbf{b}}d^2b\int \psi_f^*(\mathbf{r}_1,\ldots,\mathbf{r}_n) \times \left[e^{i\sum_{j=1}^{n}\chi_j(\mathbf{b}-\mathbf{s}_j)}-1\right]\psi_i(\mathbf{r}_1,\ldots,\mathbf{r}_n)\prod_j d^3r_j. \tag{113}$$

which is the well-known Glauber formula for high energy collisions between composite particles.

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