Lecture 1

The quest for the origin of the elements
A rotation curve is a graph of how fast something is rotating as a function of distance from the center.

Here is the rotation curve of our solar system. The farther from Sun, the slower planets are rotating.

A rotation curve is used to measure the mass inside.

But astronomers have found many galaxies with flat rotation curves, including the Milky Way.
**Flat rotation curves**

The rotation curves show that mass in a galaxy is not concentrated in the center, but is distributed throughout the galaxy.

But most of this mass cannot be seen; it gives off no light.

Observations show that up to 25% of the mass of the universe is dark (invisible) matter!

The dark matter is probably distributed throughout the galaxy halo.
Gravitational Lensing

Light passing by a massive object will bend, as for example, starlight near the Sun.

The image of a light source is distorted into a ring or an arc. The amount of bending is proportional to the mass of the object.

Gravitational lensing observations reveal the amount of mass in a galaxy.

Observation of exceptionally large gravitational lensing is a further proof that galaxies contain a large amount of dark matter.
Gravitational lensing has been observed by the Hubble Space Telescope, as seen in this picture.

**Lensing arcs** are clearly visible. They are indicative of the presence of dark matter.

The picture is from the **galaxy cluster Abell 2029**, composed of thousands of galaxies enveloped in a cloud of hot gas.
1.2 - Black Holes

Escape velocity:
Consider the kinetic energy \( K \) and gravitational potential energy \( U \) of a body close to a star of mass \( M \). Conservation of energy implies

\[
\left( K + U \right)_i = \left( K + U \right)_f \tag{1.1}
\]

The body escapes the gravitational pull of the star if it can reach infinity \( (U_f = 0) \). Then \( K_f = 0 \) because final velocity is zero and from Newton’s gravitational law

\[
\frac{1}{2} m v_e^2 - G \frac{mM}{R} = 0 \tag{1.2}
\]

where \( R \) is the initial distance between \( m \) and \( M \). From this we get the escape velocity

\[
v_e = \sqrt{\frac{2MG}{R}} \tag{1.3}
\]

\( G \) is the universal gravitational constant \( (G = 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}) \)

Examples: (a) Earth’s mass \((5.94 \times 10^{24} \text{ kg})\), \(R = 6,400 \text{ km} \rightarrow v_e = 11 \text{ km/s}\)
(b) object with \(M_\odot \) and \( R = 3 \text{ km} \rightarrow v_e > 3 \times 10^8 \text{ km/s} \) (larger than speed of light) \rightarrow \text{ black-hole}
1.3 - Luminosity and temperature

**Flux:** the total light energy emitted by one square meter of an object every second

\[ F \ (J/m^2/s = W/m^2) \] (1.4)

**Luminosity:** the total light energy emitted by the whole surface area of an object every second

\[ L = (\text{Area} \times F) \ (J/s) \] (1.5)

e.g. 100W light bulb has a surface area of about 0.01 m²

Flux = Luminosity/Area = 100 / 0.01 = 10,000 W/m²

**Radiation** – all bodies radiate EM waves at all wavelengths with a distribution of energy over the wavelengths that depends on temperature T

**Blackbody radiation:** photons are in equilibrium with the system → **Stefan-Boltzmann law:**

\[ F = \sigma T^4 \] (1.6)
**Temperature**

**Total luminosity:** integral over all wavelengths

\[ L = \int_0^\infty L_\lambda \, d\lambda \]  

(1.7)

**Effective temperature \((T_e)\):**
The temperature of a black body of the same radius as the star that would radiate the same amount of energy. Thus, from Stefan-Boltzmann law

\[ \sigma = (5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}) \]

\[ L = 4\pi R^2 \sigma T_e^4 \]  

(1.8)

**Wien’s law:** wavelength at which blackbody radiates most of its energy is given by \( \lambda_{\text{max}} = \frac{3,000,000}{T} \) as long \( \lambda_{\text{max}} \) is measured in nanometers (nm) \((1\text{nm} = 10^{-9}\text{m} = 0.000000001\text{m})\) and temperature in Kelvins

Given \( \lambda_{\text{max}} \) we can calculate \( T \) from \( T = \frac{3,000,000}{\lambda_{\text{max}}} \)

e.g. \( \lambda_{\text{max}} = 1000 \text{ nm} \) gives \( T = \frac{3,000,000}{1000} = 3,000 \text{ K} \)
1.4 - The Hertzsprung-Russell diagram

\[ M, R, L \text{ and } T \text{ do not vary independently. There are two major relationships:} \]
\[ - L \text{ with } T \]
\[ - L \text{ with } M \]

The first is known as the **Hertzsprung-Russell** (HR) diagram or the **color-magnitude** diagram.

From the Stefan-Boltzmann law

\[ L = 4\pi R^2 \sigma T_{\text{eff}}^4 \]

A star can increase luminosity by either upping the radius or the temperature. With the radius constant, the luminosity versus temperature in a log–log diagram is a straight line (**main sequence**): \( \log(L) = \text{constant} \cdot \log(T_{\text{eff}}) \).

- Stars that have the same luminosity as dimmer main sequence stars, but are to the left of them (hotter) on the HR diagram, have smaller surface areas (smaller radii).
- Bright, cool stars are very large (**Red Giants**) and lie above the main sequence line.
- Stars that are very hot and yet still dim must have small surface areas (**white dwarfs**) and lie below the main sequence.
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Stefan-Boltzmann law

$$L \propto R^2 T_e^4$$

shows that $L$ correlates with $T$

→ Hertzsprung-Russell’s idea of plotting $L$ vs. $T$ and find a path in the diagram where some information about $R$ could be found → discovery of main sequence stars (large majority of stars along the shaded band).

Typical life path of a low-mass star

Planetary nebula

Red supergiant

Horizontal branch

Red giant

Red subgiant

White dwarf

$1 M_\odot$ (Sun)

$0.3 M_\odot$

$50 M_\odot$

$10 M_\odot$

$3 M_\odot$
The Hertzsprung-Russell Diagram (HRD)
Mass-luminosity relation

For the few main-sequence stars for which masses are known, there is a
Mass-luminosity relation.

\[ L \propto M^n \]  \hspace{1cm} (1.9)

where \( n = 3 - 4 \). The slope changes at extremes, less steep for low and high mass stars.

This implies that the main-sequence (MS) on the HRD is a function of mass i.e. from bottom to top of main-sequence, stars increase in mass

Equation (1.9) only applies to MS stars with

\[ 2 < M < 20 \, M_\odot \]

and does not apply to red giants or white dwarfs.

For stars bigger than 20 \( M_\odot \), one finds

\[ L \sim M. \]
Lifetime-Mass relation

If a considerable mass fraction of a star is consumed in stellar evolution, then the lifetime of a star is given by

\[ \tau \sim \frac{M}{L} \]  
\[ (1.10) \]

\[ \tau \sim M^{-2} - M^{-3} \quad \text{for} \quad M < 20 M_\odot \]
\[ \tau \sim \text{const} \quad \text{for} \quad M \gg 20 M_\odot \]  
\[ (1.11) \]

<table>
<thead>
<tr>
<th>Mass (M_\odot)</th>
<th>Lifetime (years)</th>
</tr>
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<tbody>
<tr>
<td>60</td>
<td>3 million</td>
</tr>
<tr>
<td>30</td>
<td>11 million</td>
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<tr>
<td>10</td>
<td>32 million</td>
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<tr>
<td>3</td>
<td>370 million</td>
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<tr>
<td>1.5</td>
<td>3 billion</td>
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<tr>
<td>1</td>
<td>10 billion</td>
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<tr>
<td>0.1</td>
<td>1000s billion</td>
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\( M_\odot = \text{Sun’s mass} \)
Age and Metallicity relation

There are two other fundamental properties of stars that we can measure – age (time $t$) and chemical composition ($X, Y, Z$).

Composition parameterized with the notation:

$X = \text{mass fraction of hydrogen H}$

$Y = \text{mass fraction of helium He}$

$Z = \text{mass fraction of all other elements}$

e.g., for the Sun: $X_{\odot} = 0.747$ ; $Y_{\odot} = 0.236$ ; $Z_{\odot} = 0.017$

Note: $Z$ is often referred to as metallicity
1.5 - Hubble’s Law

Hubble found the amount of redshift depends upon the distance

- the farther away \( d \), the greater the redshift \( v \)

\[
v = H_0 \ d
\]  \hspace{1cm} (1.12)

\( H_0 \sim 70 \text{ km/s/Mpc} \)
The expansion of the Universe

- Distances between galaxies are increasing uniformly.
- There is no need for a center of the universe.

Cosmological Redshift

Universe expands \( \rightarrow \) redshift. The wavelengths get more stretched.

Size of the Universe when the light was emitted.

Size of the Universe now, when we observe the light.
Looking Back in Time

- It takes time for light to reach us: (a) $c = 300,000 \text{ km/s}$, (b) We see things “as they were” some time ago.

- The farther away, the further back in time we are looking
  - 1 billion ly means looking 1 billion years back in time.

- The greater the redshift, the further back in time
  - redshift of 0.1 is 1.4 billion ly which means we are looking 1.4 billion years into the past.

All galaxies are moving away from each other → in the past all galaxies were closer to each other.

All the way back in time, it would mean that everything started out at the same point – then began expanding.

This starting time is called the Big Bang.

The age of the Universe can be calculated using Hubble’s Law

\[
v = H_0 \, d \quad \Rightarrow \quad d = \frac{v}{H_0}
\]

But distance = velocity $\times$ time. The time is how long the expansion has been going on → The Age of the Universe)

\[
t_{\text{Universe}} = \frac{1}{H_0} \quad (1.13)
\]
As Universe expanded its temperature decreased and so did the temperature of the radiation.

This radiation should be cosmologically redshifted - mostly into microwave region - about 2.75 K

Twenty years after its prediction, it was found by Penzias and Wilson in 1964, for which they got Nobel prize. It is incredibly uniform across sky and the spectrum follows incredibly close to Planck’s blackbody radiation spectrum.

Above: how the sky looks at T=2.7 K.

Right: distribution of radiation as a function of wavelength measured by the COBE satellite compared to blackbody radiation for T=2.7 K.

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CMB Anisotropies

Shortly after the CMB was discovered one realized that there should be angular variations in temperature, as a result of density inhomogeneities in the Universe.

The denser regions cause the CMB photons to be gravitationally redshifted compared to photons arising in less dense regions.

The amplitude of the T fluctuations is roughly 1/3 of the density fluctuations.

As time passed, overdense regions became gravitationally unstable and collapsed to form galaxies, clusters of galaxies and all other structures we see in the Universe today.

From the observed CMB angular anisotropies in temperature, it is straightforward to derive what density fluctuations created them.
Figure: temperature fluctuations as measured by the satellite Wilkinson Microwave Anisotropy Probe (WMAP).

The fluctuations in temperature are at a level of $10^{-5} \, \text{T}$, and difficult to measure – first detection was in 1992.

The angular distribution of the temperature fluctuations are expanded in terms of **spherical harmonics** (any regular function of $\theta$ and $\varphi$ can be expanded in spherical harmonics)

$$
\frac{\Delta T}{T}(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}(\theta, \varphi) \quad (1.14)
$$

where the sum runs over $\ell = 1, 2, \ldots \infty$ and $m = -1, \ldots, 1$, giving $2\ell + 1$ values of $m$ for each $\ell$.

The spherical harmonics are **orthonormal functions** on the sphere, so that

$$
\int Y_{\ell m}(\theta, \varphi) Y_{\ell' m'}^*(\theta, \varphi) d\Omega = \delta_{\ell \ell'} \delta_{mm'}
$$
CMB Anisotropy

This allows us to calculate the multipole coefficients $a_{lm}$ from

$$a_{\ell m} = \int Y_{\ell m}^*(\theta, \phi) \frac{\Delta T}{T}(\theta, \phi) \, d\Omega$$

Summing over the $m$ corresponding to the same multipole number $\ell$ we have the closure relation

$$\sum_{m=-\ell}^{\ell} \left| Y_{\ell m}(\theta, \phi) \right|^2 = \frac{2\ell + 1}{4\pi}$$

Since $a_{lm}$ represent a deviation from the average temperature, their expectation value is zero, $\langle a_{lm} \rangle = 0$, and the quantity we want to calculate is the variance $\langle |a_{lm}|^2 \rangle$ to get a prediction for the typical size of the $a_{lm}$. The isotropic nature of the random process shows up in the $a_{lm}$ so that these expectation values depend only on $\ell$ not $m$. (The $\ell$ are related to the angular size of the anisotropy pattern, whereas the $m$ are related to “orientation” or “pattern”.)

The brackets $\langle \rangle$ mean an average over all observers in the Universe. The absence of a preferred direction in the Universe implies that the coefficients $\langle |a_{\ell m}|^2 \rangle$ are independent of $m$. 

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CMB Anisotropy

Since $\langle |a_{lm}|^2 \rangle$ is independent of $m$, we can define

$$C_\ell \equiv \langle |a_{\ell m}|^2 \rangle = \frac{1}{2\ell + 1} \sum_m \langle |a_{\ell m}|^2 \rangle \quad (1.15)$$

The different $a_{lm}$ are independent random variables, so that

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell m} \delta_{\ell' m'} C_\ell$$

The function $C_\ell$ (of integers $l \geq 1$) is called the angular power spectrum.

Inserting Eq. (1.15) in Eq. (1.14), one gets

$$\langle \left( \frac{\Delta T}{T} (\theta, \phi) \right)^2 \rangle = \left\langle \sum_{\ell m} a_{\ell m} Y_{\ell m}(\theta, \phi) \sum_{\ell' m'} a^*_{\ell' m'} Y^*_{\ell' m'}(\theta, \phi) \right\rangle$$

$$= \sum_{\ell' m'} \sum_{mm'} Y_{\ell m}(\theta, \phi) Y^*_{\ell' m'}(\theta, \phi) \langle a_{\ell m} a^*_{\ell' m'} \rangle$$

$$= \sum_{\ell} C_\ell \sum_{m} |Y_{\ell m}(\theta, \phi)|^2 = \sum_{\ell} \frac{2\ell + 1}{4\pi} C_\ell \approx \int \frac{\ell(\ell + 1)}{2\pi} C_\ell \, d\ln \ell$$

$$\quad (1.16)$$
\[ \frac{\ell(\ell + 1)}{2\pi} C_\ell \]

(1.17)

Thus, if we plot \((2\ell + 1)C_\ell / 4\pi\) on a linear \(\ell\) scale, or \(\ell(2\ell + 1)C_\ell / 4\pi\) on a logarithmic \(\ell\) scale, the area under the curve gives the temperature variance, i.e., the expectation value for the squared deviation from the average temperature. It has become customary to plot the angular power spectrum as \(\ell(\ell + 1)C_\ell / 2\pi\), which is neither of these, but for large \(\ell\) approximates the second case.
CMB Anisotropy

The different multipole numbers $l$ correspond to different angular scales, low $l$ to large scales and high $l$ to small scales. Examination of the functions $Y_{lm}(\theta, \phi)$ reveals that they have an oscillatory pattern on the sphere, so that there are typically $l$ “wavelengths” of oscillation around a full great circle of the sphere.

Thus the angle corresponding to this wavelength is

$$\theta_\lambda = \frac{2\pi}{l} = \frac{360^0}{l}$$

The angle corresponding to a “half-wavelength”, i.e., the separation between a neighboring minimum and maximum is then

$$\theta_{res} = \frac{\pi}{l} = \frac{180^0}{l}$$

This is the angular resolution required of the microwave detector for it to be able to resolve the angular power spectrum up to this $l$.

For example, COBE had an angular resolution of $7^0$ allowing a measurement up to $l = 180/7 = 26$, WMAP had resolution $0.23^0$ reaching to $l = 180/0.23 = 783$. 

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1.7 - Matter and Radiation Pressure in the Universe

From thermodynamics, for an isolated system,

\[ dU + dW = dQ \]  \hspace{1cm} (1.18)

where \( U \) stands for the **internal energy**, \( W \) is the work done by the system and \( Q \) is the heat transfer.

Ignoring any heat transfer, \( dQ = 0 \), and writing \( dW = Fdr = pdV \) where \( F \) is the force, \( r \) is the distance characterizing the size of the system, \( p \) is the pressure and \( V \) is the volume, then

\[ dU = -pdV \]  \hspace{1cm} (1.19)

Denoting the **energy density** by \( \rho \):

\[ U = \rho V \]  \hspace{1cm} (1.20)

\[ \frac{dU}{dt} = \frac{d\rho}{dt} V + \rho \frac{dV}{dt} = -p \frac{dV}{dt} \]  \hspace{1cm} (1.21)

Since \( V \propto r^3 \), then \( (dV/dt)/V = 3(dr/dt)/r \).

Thus,

\[ \frac{d\rho}{dt} = -3(\rho + p) \frac{1}{r} \frac{dr}{dt} \]  \hspace{1cm} (1.22)
Matter Dominated Universe

Assuming all energy to be in the form of matter, the relation between matter, $M$, density, $\rho$, and radius, $r$, means that

$$\rho = \frac{M}{4\pi r^3 / 3} \quad (1.23)$$

Comparing Eqs. (1.22) and (1.24) we conclude that

$$\frac{d\rho}{dt} = \frac{d\rho}{dr} \frac{dr}{dt} = -3\rho \frac{1}{r} \frac{dr}{dt} \quad (1.24)$$

That is, if no kinetic energy is taken into account, pressure is zero for a system with mass $M$. This is the same pressure as in the ideal gas law for zero temperature.

$p = 0 \quad (1.30)$

for matter dominated Universe.
Radiation Dominated Universe

Let us consider radiation modes in a cavity based on analogy with a string held fixed at two points separated by a distance $L$. The possible wavelengths, $\lambda$, of a standing wave on the string obey the relation

$$L = \frac{n\lambda}{2}$$  \hspace{1cm} (1.31)

$n = 1, 2, 3, \ldots$. Radiation travels at the velocity of light, so that

$$c = f\lambda = f \frac{2L}{n}$$  \hspace{1cm} (1.32)

where $f$ is the frequency. Planck's formula for the energy of a quantum of radiation with frequency $f = \omega/2\pi$ is $U = \hbar \omega = hf$, where $\hbar = h/2\pi$, and $h$ is Planck's constant. Thus,

$$U = \frac{1}{L} \frac{n\hbar c}{2} \sim V^{-1/3}$$  \hspace{1cm} (1.33)

where $V = L^3$ is the volume of a cube of length $L$. Using Eq. (1.19) the pressure becomes

$$p = -\frac{dU}{dV} = \frac{1}{3} \frac{U}{V}$$  \hspace{1cm} (1.34)

This, together with $\rho = U/V$, yields the radiation pressure:

$$p = \frac{\gamma \rho}{3}$$  \hspace{1cm} (1.35)

with $\gamma = 1$ (radiation).

For matter pressure $\gamma = 0$. 

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### 1.8 - Friedmann Equation

**Birkoff's Theorem:** the gravitational force on the particle inside a uniform shell is the same as if the enclosed mass within radius $r$ is localized entirely at the origin, $r = 0$.

If it is located a distance $r$ from the center of the dust, the total energy $E$ of the particle is then given by

$$E = T + V = \frac{1}{2} m \dot{r}^2 - G \frac{Mm}{r} = \frac{1}{2} m \dot{r}^2 \left( H^2 - \frac{8\pi G}{3} \rho \right)$$

(1.36)

Where $\dot{r} = dr/dt$, $H = (dr/dt)/r$ is the Hubble constant (see Eq. 1.12), $G$ is Newton's constant, and we used $M = \rho 4\pi r^3/3$ in the last passage.

Using Eq. (1.3) for the escape velocity, $v_{esc} = [2GM/r]^{1/2} = [(8\pi G/3)pr^2]^{1/2}$, Eq. (1.36) can be written as

$$\dot{r}^2 = v_{esc}^2 - k'$$

(1.37)

with $k' = -2E/m$. The constant $k'$ can either be negative, zero or positive, corresponding to the total energy $E$ being positive, zero or negative.
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**Friedmann Equation**

Equation (1.36) is re-arranged as

\[
H^2 = \frac{8\pi G}{3} \rho + \frac{2E}{mr^2}
\]  

(1.38)

Writing the distance in terms of a scale factor \(a\) and a constant length \(s\) as \(r(t) = a(t)s\), and defining \(k = k'/s^2 = -2E/ms^2\), it follows that

\[
H^2 = \left(\frac{\ddot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2}
\]  

(1.39)

since \(\dot{r}/r = \dot{a}/a\) and \(\ddot{r}/r = \ddot{a}/a\). This is the **Friedmann equation**. It specifies the speed of recession of the Universe.

Even though Friedmann equation was derived for matter, it is also true for radiation. Exactly the same equation is obtained from the **general relativity Einstein field equations**.

The factor \(k\) can be rescaled so that instead of being negative, zero or positive it takes on the values \(-1, 0\) or \(+1\). In Newtonian mechanics this corresponds to unbound, critical or bound trajectories. From a geometric point of view, this corresponds to an **open, flat or closed Universe**.
1.9 - The Cosmological constant

The acceleration for the Universe is obtained from Newton's second equation, i.e.

$$-G \frac{mm}{r^2} = m\ddot{r}$$  \hspace{1cm} (1.40)

In terms of the density and the scale \(a\),

$$\frac{F}{mr} = \frac{\ddot{r}}{r} = \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \rho$$  \hspace{1cm} (1.41)

Another way to get this equation is to take the time derivative of Eq. (1.39) (valid for matter and radiation and \(k=0\))

$$\frac{d}{dt} \dot{a}^2 = 2\dddot{a} = \frac{8\pi G}{3} \frac{d}{dt} \left( \rho a^2 \right)$$  \hspace{1cm} (1.42)

Upon using Eq. (1.22) the acceleration equation is obtained as

$$\dddot{a} = -\frac{4\pi G}{3} (\rho + 3p) = -\frac{4\pi G}{3} (1 + \gamma) \rho$$  \hspace{1cm} (1.43)

which reduces to Eq. (1.41) for the matter equation of state (\(\gamma = 0\)).
The Cosmological constant

The Universe is unstable to gravitational collapse. Both Newton and Einstein believed that the Universe is static. In order to obtain this Einstein introduced a repulsive gravitational force, called the cosmological constant. In order to obtain a possibly zero acceleration, a positive term (conventionally taken as $\Lambda/3$) is added to the acceleration Eq. (1.43) as

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}$$

which, with the proper choice of $\Lambda$, can give the required zero acceleration for a static Universe.

The question now is how this repulsive force enters the Friedmann Equation. Identifying the force from

$$\ddot{r} = \ddot{a} = \frac{F_{\text{exp}}}{mr} = \frac{\Lambda}{3}$$

and using

$$F_{\text{exp}} = \frac{\Lambda}{3} mr = -\frac{dV_{\text{exp}}}{dr}$$

gives the potential energy

$$V_{\text{exp}} = -\frac{1}{2} \frac{\Lambda}{3} mr^2$$

which is just a simple repulsive harmonic oscillator.
The Cosmological constant

Replacing this into the conservation of energy equation,

\[ E = T + V = \frac{1}{2} m \dot{r}^2 - G \frac{M m}{r} - \frac{1}{2} \frac{\Lambda}{3} m r^2 = \frac{1}{2} m r^2 \left( H^2 - \frac{8\pi G}{3} \rho - \frac{\Lambda}{3} \right) \]  

(1.46)

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \]  

(1.47)

Equations (1.44) and (1.47) constitute the fundamental equations of motion that are used in all discussions of Friedmann models of the Universe.

One often writes the cosmological constant in terms of a vacuum energy density as \( \Lambda = 8\pi G \rho_{\text{vac}} \) so that the velocity and acceleration equations become

\[ H^2 = \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \left( \rho + \rho_{\text{vac}} \right) - \frac{k}{a^2} \]  

(1.48)

and

\[ \frac{\ddot{a}}{a} = - \frac{4\pi G}{3} \left( 1 + \gamma \right) \rho + \frac{8\pi G}{3} \rho_{\text{vac}} \]  

(1.49)
From Eq. (1.47),
\[ 1 = \frac{8\pi G \rho}{3H^2} - \frac{k}{H^2 a^2} + \frac{\Lambda}{3H^2} \]  
\[ (1.50) \]

Each of the terms in this equation has special significance. The mass density is
\[ \Omega_m = \frac{8\pi G \rho}{3H^2} \]  
\[ (1.51) \]

The curvature density is
\[ \Omega_k = -\frac{k}{H^2 a^2} \]  
\[ (1.52) \]

The vacuum energy density, or dark energy, is
\[ \Omega_\Lambda = \frac{\Lambda}{3H^2} \]  
\[ (1.53) \]

Another quantity of interest is the critical density, given by
\[ \rho_{\text{crit}} = \frac{3H^2}{8\pi G} \]  
\[ (1.54) \]

in terms of which the mass density can be written as \( \Omega_m = \rho/\rho_{\text{crit}} \). In terms of the present value of the Hubble parameter the critical density is,
\[ \rho_{\text{crit}} = 1.88 \times 10^{-20} \, h^2 \, \text{g cm}^{-3} \]  
\[ (1.55) \]

\[ h = \frac{H}{100 \, \text{Mpc}^{-1} \, \text{s}^{-1}} \approx 0.7 \]  
\[ (1.56) \]
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**Dark Energy**

Defining

\[ \Omega = \Omega_m + \Omega_\Lambda \]  \hspace{1cm} (1.57)

the Friedmann equation can be rewritten as

\[ (\Omega - 1)H^2 = \frac{k}{a^2} \]  \hspace{1cm} (1.58)

so that \( k = 0, \ +1, \ -1 \) corresponds to \( \Omega = 1, \ \Omega > 1 \) and \( \Omega < 1 \).

The matter density decreases with the radius of the Universe as \( \rho (t = 0)/\rho(t) = a^3/a_0^3 \). Thus, we can write a **mixture of matter and dark energy** by (here, the index “0” means the present value of the variables.)

\[ \rho = \rho_m + \rho_\Lambda = \rho_m(a_0/a)^3 + \rho_\Lambda \]  \hspace{1cm} (1.59)

and the Friedmann equation becomes

\[ \left( \frac{\dot{a}}{a} \right)^2 - H^2 \Omega_m(a_0/a)^2 - H^2 \Omega_\Lambda_0 = - \frac{k}{a^2} \]  \hspace{1cm} (1.60)

Using \( k = 0 \) (flat Universe), \( \Omega_m(a_0/a)^2 = 1 - \Omega_\Lambda_0 \) and, for simplicity \( a_0 = 1 \) (in appropriate units), we get

\[ \left( \frac{\dot{a}}{a} \right)^2 = H_0^2 \left[ (1 - \Omega_\Lambda_0) \frac{1}{a^3} + \Omega_\Lambda_0 \right] \]
Dark Energy

Integrating the last equation over time, with \( t_0 \) denoting the age of the Universe, we get

\[
H_0 t_0 = \int da \frac{\sqrt{a}}{\sqrt{1 - \Omega_{\Lambda_0} + \Omega_{\Lambda_0} a^2}} = \frac{2}{3\sqrt{\Omega_{\Lambda_0}}} \ln \left( \frac{1 + \sqrt{\Omega_{\Lambda_0}}}{\sqrt{1 - \Omega_{\Lambda_0}}} \right)
\]

where \( H_0 \) is the present value of the Hubble constant. We thus see that, as \( \Omega_{\Lambda_0} \to 1 \), then \( t_0 \to \infty \). It is thus necessary to have some matter to keep the age of the Universe finite.

We can turn this argument around. Assuming the age of the Universe to be \( t_0 = 13.7 \text{ Gy} \) we get \( \Omega_{\Lambda_0} = 0.72 \), or \( \Omega_{m_0} = 0.28 \), i.e. only 28% of the Universe is matter and 72% is dark energy. Observations also indicate that only 4% of the Universe is baryonic (normal) matter, and that the remaining 24% is in some other still unknown form, a dark matter.

Dark matter and dark energy thus compose about 95% of the Universe.
1 - Calculate the escape velocity from a particle ejected from the Sun.

2 – (a) According to **Wien’s law**, the peak of the energy density distribution of a black-body at

\[
f_{\text{peak}} \approx 2.8 \frac{k_B T}{h} \tag{1.62}
\]

implies that \( f_{\text{peak}} / T \) is a constant. Evaluate this constant in SI units.

(b) The Sun radiates approximately as a black-body with \( T \sim 5800 \) K. Compute \( f_{\text{peak}} \) for solar radiation. Where in the electromagnetic spectrum does the peak emission lie?

3 - The cosmic microwave background has a black-body spectrum at a temperature of 2.725 K. Repeat the previous problem to find the peak frequency of its emission, and also find the corresponding wavelength.
4 - Suppose that the Universe were full of baseballs, each of mass \( m = 0.145 \) kg and radius \( r = 0.0369 \) m. If the baseballs were distributed uniformly throughout the Universe, what number density of baseballs would be required to make the density equal to the critical density?

5 - Suppose you are in an infinitely large, infinitely old Universe in which the average density of stars is \( n_{\text{stars}} = 10^9 \text{ Mpc}^{-3} \) and the average stellar radius is equal to the Sun’s radius: \( R_{\text{sun}} = R = 7 \times 10^8 \text{ m} \). How far, on average, could you see in any direction before your line of sight struck a star (Olber’s paradox)? (Hint: remember that a molecule in a gas travels an average distance \( \lambda = 1/n\sigma \), where \( \lambda \) is the “mean free path”, \( n \) is the number density of molecules and \( \sigma \) is the cross sectional area of each molecule).