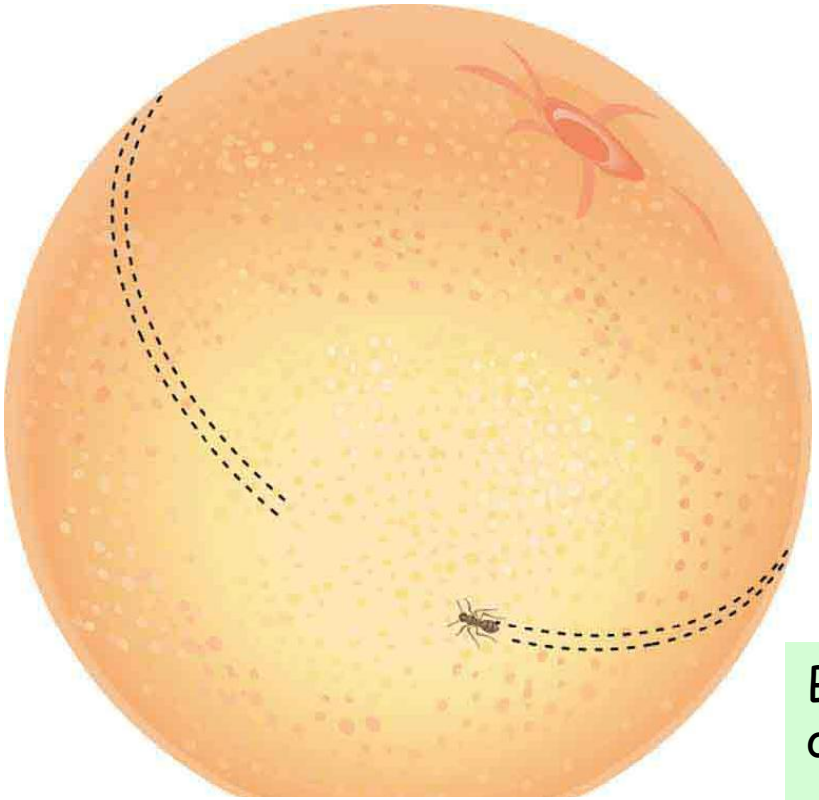
Lecture 2

The quest for the origin of the elements

2.1 - Geometry of the Universe



2-dimensional analogy: Surface of a sphere.

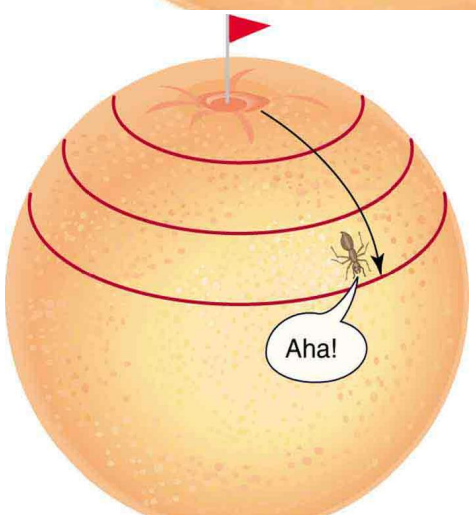
The surface is finite, but has no edge.

For a creature living on the sphere, having **no sense of the third dimension**, there is no center (on the sphere). All points are equal.

Any point on the surface can be defined as the center of a coordinate system.

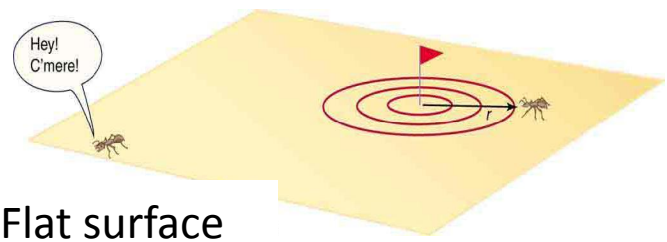
But, how can a 2-D creature investigate the geometry of the sphere?

Answer: **Measure curvature of its space.**



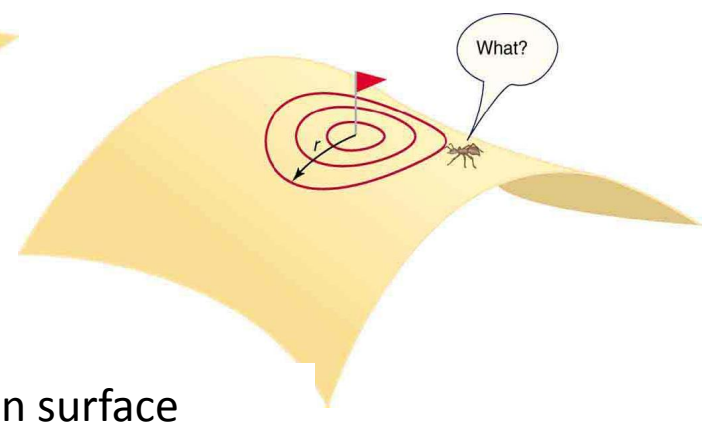
Closed surface

(positive curvature, $k = 1$)



Flat surface

(zero curvature, $k = 0$)



Open surface

(negative curvature, $k = -1$)

Geometry of the Universe

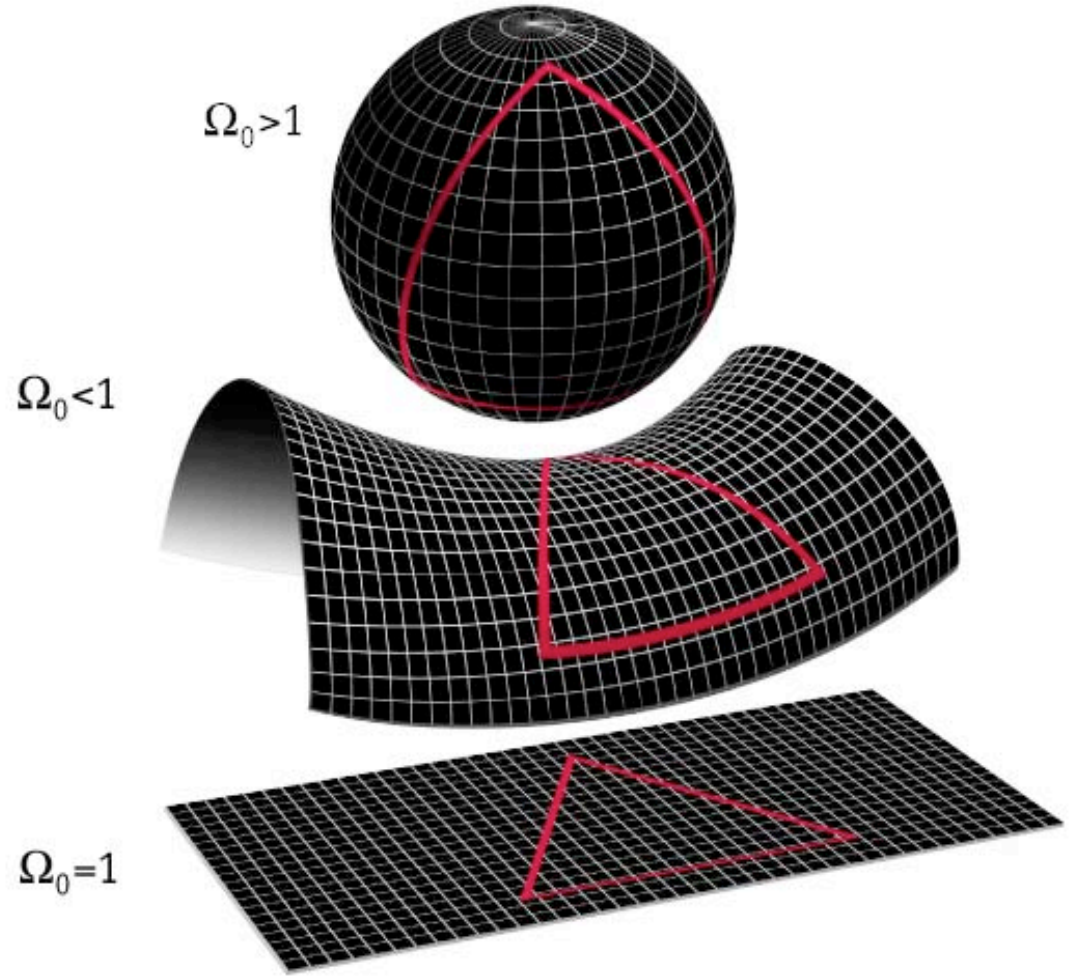
These are the three possible geometries of the Universe:

closed, **open** and **flat**, corresponding to a density parameter $\Omega_m = \rho/\rho_{crit}$ which is greater than, less than or equal to 1.

The relation to the curvature parameter is given by Eq. (1.58).

The closed universe is of finite size. Traveling far enough in one direction will lead back to one's starting point.

The open and flat universes are infinite and traveling in a constant direction will never lead to the same point.



2.2 - Static Universe

The static Universe requires $a = a_0$ = constant and thus $d^2a/dt^2 = da/dt = 0$. From Eq. (1.44), $d^2a/dt^2 = 0$ requires that

$$\Lambda = 4\pi G\left(\rho + 3p\right) = 4\pi G\left(1+\gamma\right)\rho \qquad (2.1)$$

If there is no cosmological constant ($\Lambda = 0$) then either $\rho = 0$ which is an empty Universe, or $p = -\rho/3$ which requires negative pressure. Both of these alternatives were unacceptable to Einstein and therefore he concluded that a cosmological constant was present, i.e. $\Lambda \neq 0$. From Equation (2.1) this implies

$$\rho = \frac{\Lambda}{4\pi G\left(1+\gamma\right)} \qquad (2.2)$$

and because ρ is positive this requires a positive Λ. Inserting Eq. (2.2) into Eq. (1.47), it follows that

$$\Lambda = \frac{3\left(1+\gamma\right)}{3+\gamma}\left[\left(\frac{\dot{a}}{a_0}\right)^2 + \frac{k}{a_0^2}\right] \qquad (2.3)$$

Now imposing $da/dt = 0$ and assuming a matter equation of state ($\gamma = 0$) implies $\Lambda = k/a^2_0$. However the requirement that Λ be positive forces $k = +1$, giving

$$\Lambda = \frac{1}{a_0^2} = \text{constant} \qquad (2.4)$$

Static Universe

Thus the cosmological constant for the static Universe is not any value but rather simply the inverse of the scale factor squared, where the scale factor has a fixed value in this static model.

Using Eqs. (2.2) and (2.4) we obtain that the static Universe is closed with the scale factor (which in this case gives the radius of curvature) given by (**Einstein radius**)

$$a_0 = \frac{1}{\sqrt{4\pi G \rho_0}} \qquad (2.5)$$

Using $\rho_0 = \rho_{crit}$ the numerical value of Einstein radius is of order of 10^{10} light years.

It is worth noting that even though the model is static, it is unstable:

if perturbed away from the equilibrium radius, the Universe will either expand to infinity or collapse. If we increase a from a_0, then the Λ term will dominate the equations, causing a runaway expansion, whereas if we decrease a from a_0, the dust (matter) term will dominate, causing collapse. Therefore, this model is also physically unsound, and this is a far worse problem than the (to Einstein) unattractive presence of Λ.

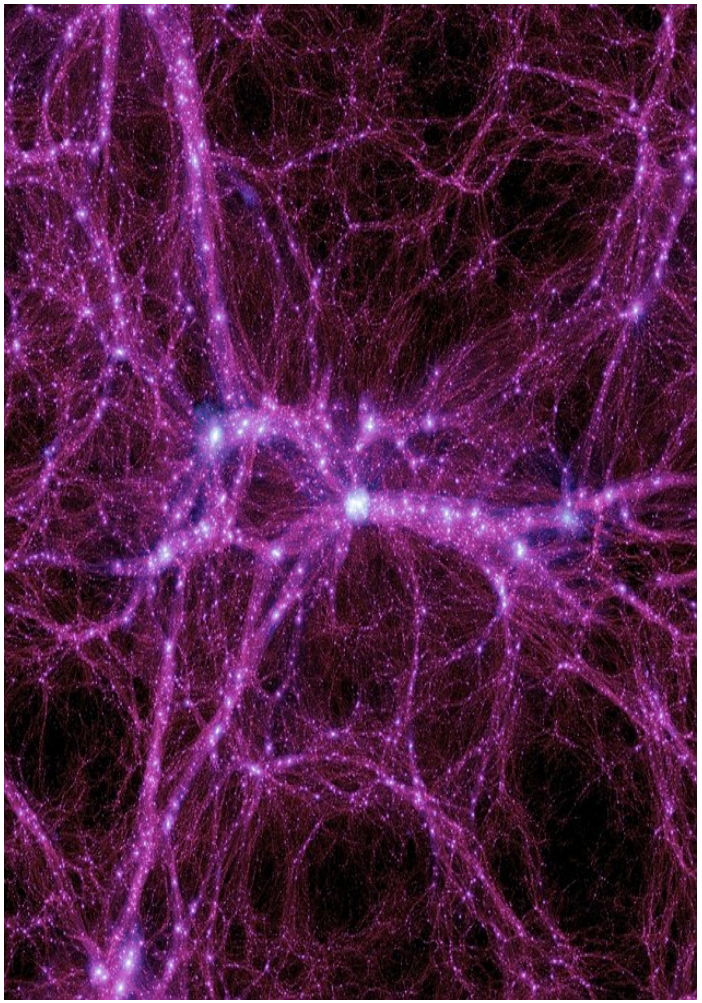
2.3 - Matter and Radiation Universes

Equation (1.22) can be rewritten as

$$\dot{\rho} + 3\left(p + \rho\right)\frac{\dot{a}}{a} = 0 \qquad (2.6)$$

or

$$\frac{d}{dt}\left(\rho a^3\right) + p\frac{da^3}{dt} = 0 \qquad (2.7)$$



From Eq. (1.35), $p = \gamma\rho/3$, from which follows that

$$\frac{d}{dt}\left(\rho a^{3+\gamma}\right) = 0 \qquad (2.8)$$

Integrating this we obtain

$$\rho = \frac{c}{a^{3+\gamma}} \qquad (2.9)$$

where c is a constant. This shows that the density falls as a^{-3} for matter dominated and a^{-4} for radiation-dominated Universes.

Matter and Radiation Universes

We consider here a flat, $k = 0$, Universe. Currently the Universe is in a matter dominated phase whereby the dominant contribution to the energy density is due to matter. However the early Universe was radiation dominated and the very early Universe was vacuum dominated. With $k = 0$, there will only be one term on the right hand side of Eq. (1.49) depending on what is dominating the Universe.

For a matter ($\gamma = 0$) or radiation ($\gamma = 1$) dominated Universe the right hand side of Eq. (1.49) will be of the form $1/a^{3+\gamma}$ (ignoring the vacuum energy), whereas for a vacuum dominated Universe the right hand side will be a constant. The solution to the Friedmann equation (1.39) for a **radiation dominated Universe** will thus be (from $a da \propto dt$)

$$a \approx t^{1/2} \qquad (2.10)$$

while for the **matter dominated** case it will be (from $a^{1/2} da \propto dt$)

$$a \approx t^{2/3} \qquad (2.11)$$

One can see from d^2a/dt^2 that these results give negative acceleration, corresponding to a decelerating expanding Universe.

2.4 - Summary: Solutions of Friedmann Equation with k=0

Friedmann equation for $k = 0$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = \frac{8\pi G}{3}\rho \qquad (2.12)$$

To solve one needs $\rho(a)$. There are two important cases:

- $\rho \sim 1/a^4$ (**radiation-dominated Universe**). Then

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{1/2}; \quad \rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2} \qquad (2.13)$$

- $\rho \sim 1/a^3$ (**matter-dominated Universe**). Then

$$\frac{a(t)}{a_0} = \left(\frac{t}{t_0}\right)^{2/3}; \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} \qquad (2.14)$$

Here the index '0' refers to the values today. As usual, we have set $a(t_0) = 1$.

2.5 - Decceleration parameter

Deceleration parameter of the Universe (definition):

$$q = -\frac{\ddot{a}(t)}{H^2 a(t)} \qquad (2.15)$$

If the Universe is matter dominated, then Eq. (1.49) (with $\rho_{vac} = 0$) yields

$$\rho = \frac{3H^2}{4\pi G} q \qquad (2.16)$$

Plugging this result into the Friedmann Equation (1.39), one gets

$$-k = a^2 H^2 (1 - 2q) \qquad (2.17)$$

Since both $da/dt = 0$ and $H = 0$, for **flat Universe** ($k = 0$) we get $q = 1/2$. When combined with Eq. (2.16), this yields the critical density, Eq. (1.54), the density needed to yield the flat Universe. We also get $q > 1/2$ if $k = 1$ and $q < 1/2$ if $k = -1$.

The quantity q provides the relationship between the density of the Universe and the critical density,

$$q = \frac{\rho}{2\rho_{crit}} \qquad (2.18)$$

2.6 - General Solutions of Friedmann Equation

The **Friedmann equation** is given by

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3}} \qquad (2.19)$$

Using $\Lambda = 8\pi G\rho_{vac}$ it can also be written as

$$\boxed{H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho + \rho_{vac}\right) - \frac{k}{a^2}} \qquad (2.20)$$

Using Newton's 2nd law, the Gravitational law, and $p = \gamma\rho/3$ for matter $\gamma = 0$ or radiation $\gamma = 1$, one can also rewrite it as

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(1+\gamma\right)\rho + \frac{8\pi G}{3}\rho_{vac}} \qquad (2.21)$$

We now will discuss the applications of these equations to the **structure** and **dynamics** of the **Universe**.

2.7 - Flat Universe

In this case, the Universe has no curvature, $k = 0$ (and using $\Lambda = 0$), and Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad (2.22)$$

As we saw in Eq. (2.17) $q = 1/2$. We consider the matter-dominated and the radiation-dominated Universes separately.

2.7.1 - Matter dominated Universe

We have $p = 0$ and $a^3\rho = const$ and, using Eq. (2.19) (with $\Lambda = 0$), we get

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a_0}{a}\right)^3 \qquad (2.23)$$

which leads to

$$\int a^{1/2}\,da = \frac{2}{3}a^{3/2} + C = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\;t \qquad (2.24)$$

Flat, matter dominated Universe

At the Big Bang, $t = 0$, $a = 0$, so $C = 0$. Since the Universe is assumed flat, $k = 0$, $\rho_0 = \rho_{crit}$, we have

$$\frac{a}{a_0} = \left(6\pi G\rho_0\right)^{1/3} t^{2/3} = \left(6\pi G\rho_{crit}\right)^{1/3} t^{2/3} = \left(6\pi G \frac{3H_0^2}{8\pi G}\right)^{1/3} t^{2/3}$$

or

$$\boxed{\frac{a}{a_0} = \left(\frac{3H_0}{2}\right)^{2/3} t^{2/3}} \qquad (2.25)$$

From this we compute the age of the Universe t_0, which corresponds to the Hubble rate H_0 and the scale factor $a = a_0$, to be

$$\boxed{t_0 \approx 9.1 \times 10^9 \text{ years}} \qquad (2.26)$$

Notice that we had already obtained the correct power dependence of a as a function of t in Eq. (2.11). Now we also obtained the correct multiplicative coefficient in Eq. (2.25).

The model described in this subsection in known as the **Einstein–de Sitter model**.

2.7.2 - Flat, Radiation dominated Universe

We have $p = \rho/3$ and $a^4\rho = const$ and, using Eq. (2.19) (with $\Lambda = 0$), we get

$$\int a\,da = 2a^2 + C = \sqrt{\frac{8\pi G\rho_0 a_0^3}{3}}\,t \qquad (2.27)$$

Again, at the Big Bang, $t = 0$, $a = 0$, thus $C = 0$, and ρ_0 is equal to the critical density ρ_{crit}. Therefore,

$$\frac{a}{a_0} = \left(\frac{2\pi}{3}G\rho_{crit}\right)^{1/4} t^{1/2} = \left(\frac{H_0}{2}\right)^{1/2} t^{1/2} \qquad (2.28)$$

For a radiation-dominated Universe, the age of the Universe would be much longer than for a matter-dominated Universe:

$$t_0 \approx 27 \times 10^9 \text{ years} \qquad (2.29)$$

Notice that we had already obtained the correct power dependence of a as a function of t in Eq. (2.25). Now we also obtained the correct multiplicative coefficient in Eq. (2.29).

2.8 - Closed Universe

In this case, we have $k = 1$ and $q > \frac{1}{2}$. Here we only present the results.

2.8.1 - Matter dominated

We have $p = 0$ and $a^3\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \frac{q}{2q-1}\left(1-\cos\eta\right), \qquad \text{with} \qquad \frac{t}{t_0} = \frac{q}{2q-1}\left(\eta-\sin\eta\right) \tag{2.30}$$

where q is the decceleration parameter, given by Eq. (2.15).

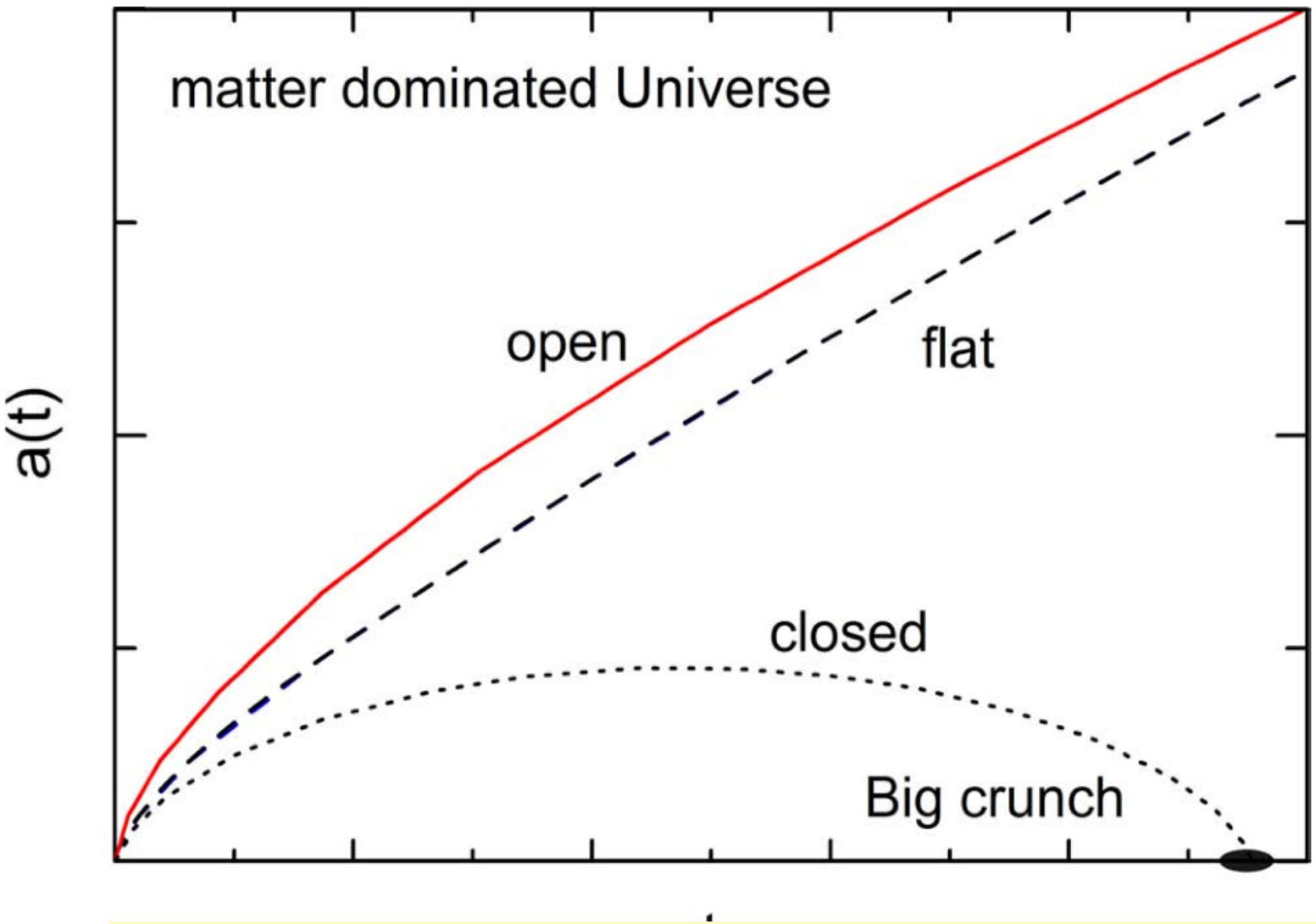
2.8.2 - Radiation dominated

We have $p = \rho/3$ and $a^4\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \sqrt{\frac{2q}{2q-1}}\sin\eta, \qquad \text{with} \qquad \frac{t}{t_0} = \sqrt{\frac{2q}{2q-1}}\left(1-\cos\eta\right) \tag{2.31}$$

In both matter- and radiation-dominated closed Universes, the evolution is cyclic — the scale factor grows at an ever-decreasing rate until it reaches a point at which the expansion is halted and reversed. The Universe then starts to compress and it finally collapses in the **Big Crunch**. (see next figure.)

Matter Dominated Universe



Evolution of the scale factor *a(t)* for the flat, closed and open matter-dominated Friedmann Universe.

2.9 - Open Universe

In this case, we have $k = -1$ and $q < \frac{1}{2}$. Here we only present the results.

2.9.1 - Matter dominated

We have $p = 0$ and $a^3\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \frac{q}{2q-1}\left(\cosh\eta - 1\right), \qquad \text{with} \qquad \frac{t}{t_0} = \frac{q}{2q-1}\left(\sinh\eta - 1\right) \tag{2.32}$$

2.9.2 - Radiation dominated

We have $p = \rho/3$ and $a^4\rho = const$. The solutions are obtained in parametric form:

$$\frac{a}{a_0} = \sqrt{\frac{2q}{1-2q}}\sinh\eta, \qquad \text{with} \qquad \frac{t}{t_0} = \sqrt{\frac{2q}{1-2q}}\left(\cosh\eta - 1\right) \tag{2.33}$$

The previous figure summarizes the evolution of the scale factor a(t) for open, flat and closed matter-dominated Universes. In earlier times, one can expand the trigonometric and hyperbolic functions to leading terms in powers of η, and the a and *t* dependence on *η* for the different curvatures are given in the next table. This shows that at early times the curvature of the Universe does not matter. The singular behavior at early times is essentially independent of the curvature of the Universe, or k. **The Big Bang is a matter dominated singularity.**

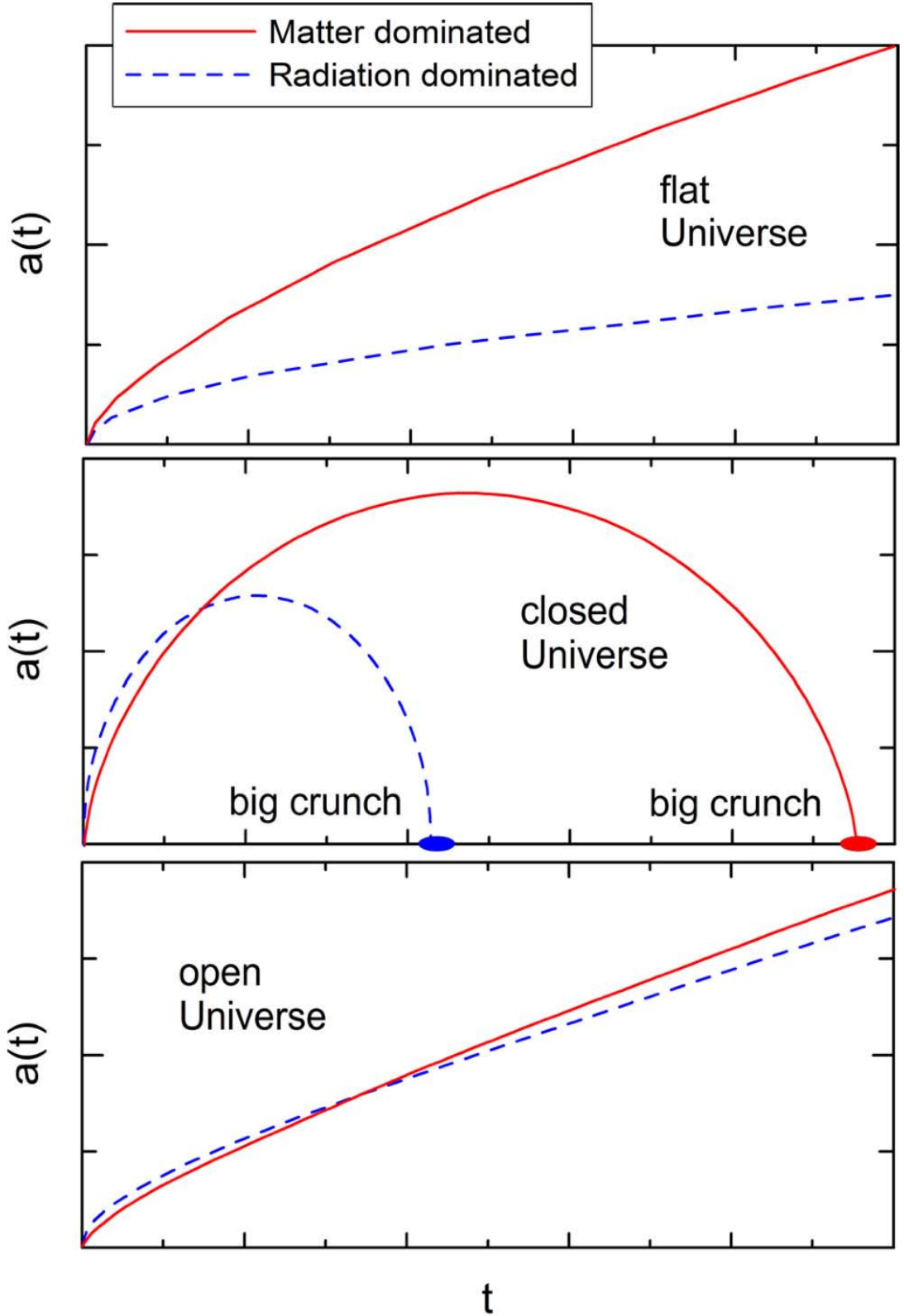
Matter Dominated Universe

Curvature	All η		Small η		
k	a/a_0	t	a	t	$a(t)$
0	$(6\pi G\rho_0)^{1/3}t^{2/3}$	—	$\propto t^{2/3}$	—	$\propto t^{2/3}$
1	$\frac{q}{2q-1}(1-\cos\eta)$	$\frac{q}{2q-1}(\eta-\sin\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$
-1	$\frac{q}{1-2q}(\cosh\,\eta-1)$	$\frac{q}{1-2q}(\sinh\,\eta-\eta)$	$\propto \eta^2$	$\propto \eta^3$	$\propto t^{2/3}$

Solutions of Friedmann equation for matter-dominated Universe.

Summary: Solutions of Friedmann Equation

Evolution of the scale factor a(t) for the flat (upper panel), closed (middle panel) and open (lower panel) Friedmann Universe.



2.10 - Flatness problem

According to Eq. (1.58), the density parameter satisfies an equation of the form

$$\left|\Omega - 1\right| = \frac{\left|\rho(t) - \rho_c\right|}{\rho_c} = \frac{1}{\dot{a}^2(t)} \qquad (2.34)$$

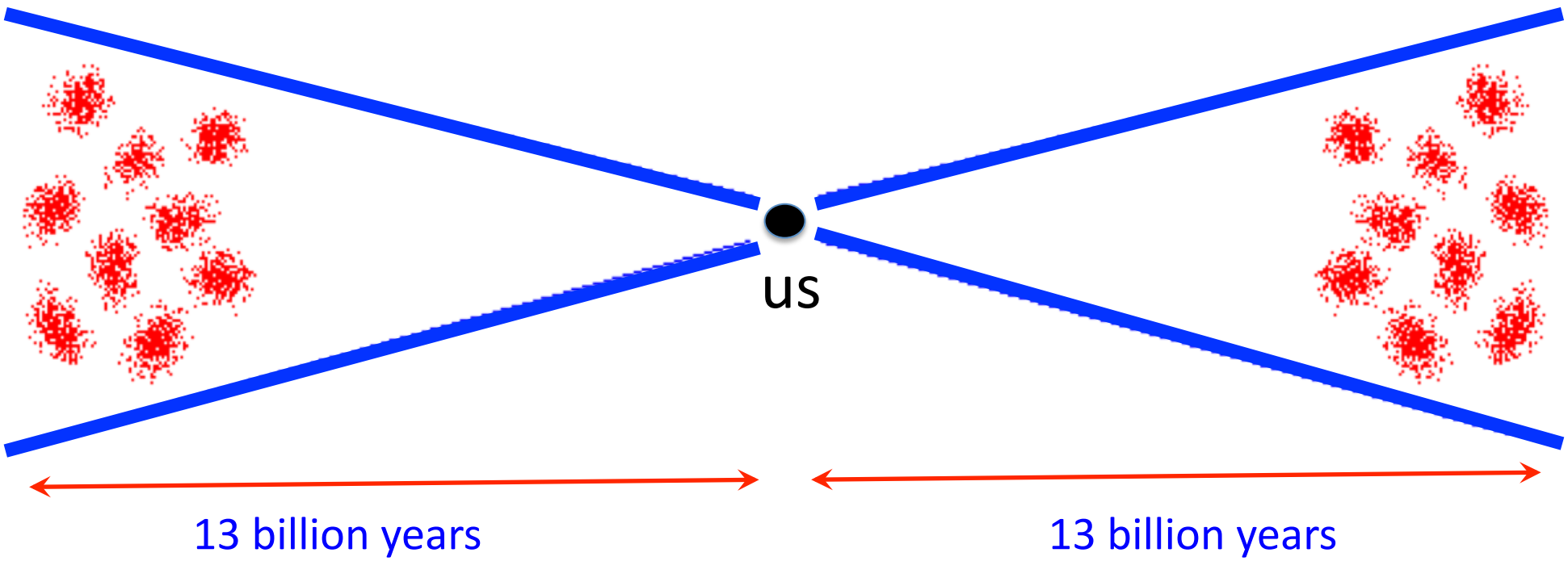
The present day value of Ω is known only roughly, $0.1 \leq \Omega \leq 2$. On the other hand $(da/dt)^2 \sim 1/t^2$ in the early stages of the evolution of the Universe, so the quantity $|\Omega - 1|$ was extremely small. One can show that in order for Ω to lie in the range $0.1 \leq \Omega \leq 2$ now, the early Universe must have had

$$\left|\Omega - 1\right| = \frac{\left|\rho(t) - \rho_c\right|}{\rho_c} \leq 10^{-59} \qquad (2.35)$$

This means that if the density of the Universe were initially greater than ρ_c, say by $10^{-55}\rho_c$, it would be closed and the Universe would have collapsed long time ago. If on the other hand the initial density were $10^{-55}\rho_c$ less than ρ_c, the present energy density in the Universe would be vanishingly low and the life could not exist. The question of why the energy density in the early Universe was so fantastically close to the critical density is usually known as the **flatness problem**.

2.11 - Horizon problem

The number and size of density fluctuations on both sides of the sky are similar, yet they are separated by a distance that is greater than the speed of light times the age of the Universe, i.e. they should have no knowledge of each other by special relativity (c, the speed of light is finite: c = 300,000 km/s).



At some time in the early Universe, all parts of spacetime were causally connected, this must have happened after the spacetime "foam" era, and before the time where thermalization of matter occurred. This is known as the **horizon problem**.

2.12 - Inflation

Inflation occurs when the vacuum energy contribution dominates the ordinary density and curvature terms in Friedmann Equation (2.19). Assuming these are negligible and substituting Λ = constant, one gets

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2(t) = +\frac{\Lambda}{3} \qquad (2.36)$$

The solution is

$$a(t) = \exp\left\{\sqrt{\frac{\Lambda}{3}}t\right\} \qquad (2.37)$$

When the Universe is dominated by a cosmological constant, the **expansion rate grows exponentially**.

Inflation phase

Particle physics predicts that the interactions among particles in the early Universe are controlled by forces (or force fields (sf)), which can have negative pressure ($p_{sf} = -\rho_{sf}$, where ρ_{sf} is the energy density of the fields.)

If $\rho_{sf} \gg \rho_{rad}$, Friedmann equation reads:

$$\frac{d^2a}{dt^2} = -\frac{4\pi G}{3}\left[\rho_{sf} + 3p_{sf}\right]a = \frac{8\pi G}{3}\rho_{sf}a \qquad (2.38)$$

The solution is:

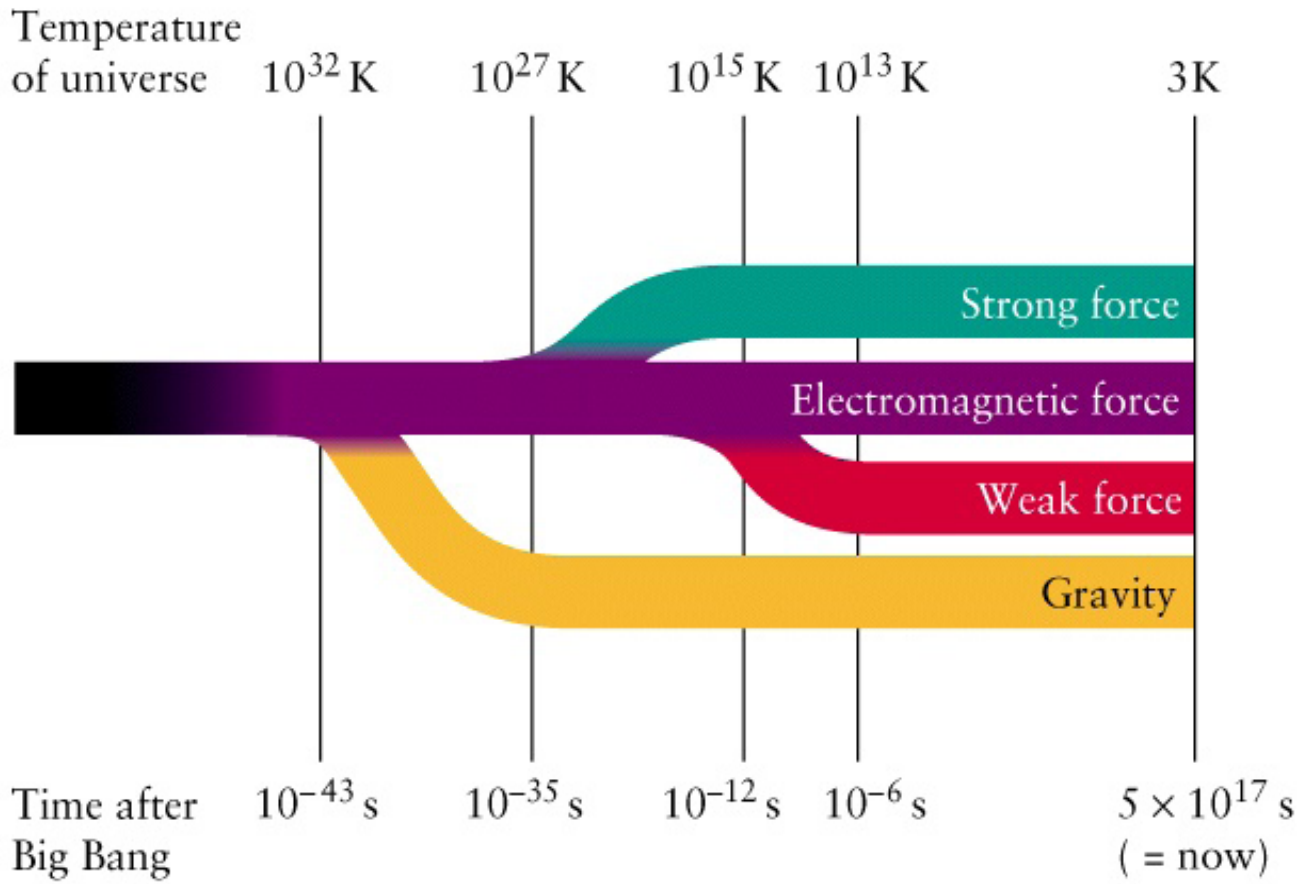
$$a(t) = a_0 \exp\left\{\frac{t}{t_{int}}\right\}; \qquad t_{int} = \sqrt{\frac{3}{8\pi G\rho_{sf}}} \qquad (2.39)$$

The model predicts $t_{inf} \sim 10^{-34}$ s. Inflation lasts as long as the energy density associated with the force fields is transformed into expansion of the Universe. If this lasts from $t_1 = 10^{-34}$ s to $t_2 = 10^{-32}$ s, then the Universe expands by a factor $exp(\Delta t/t_{inf}) \sim e^{100} \sim 10^{43}$.

After this exponential increase, the Universe develops as we discussed before.

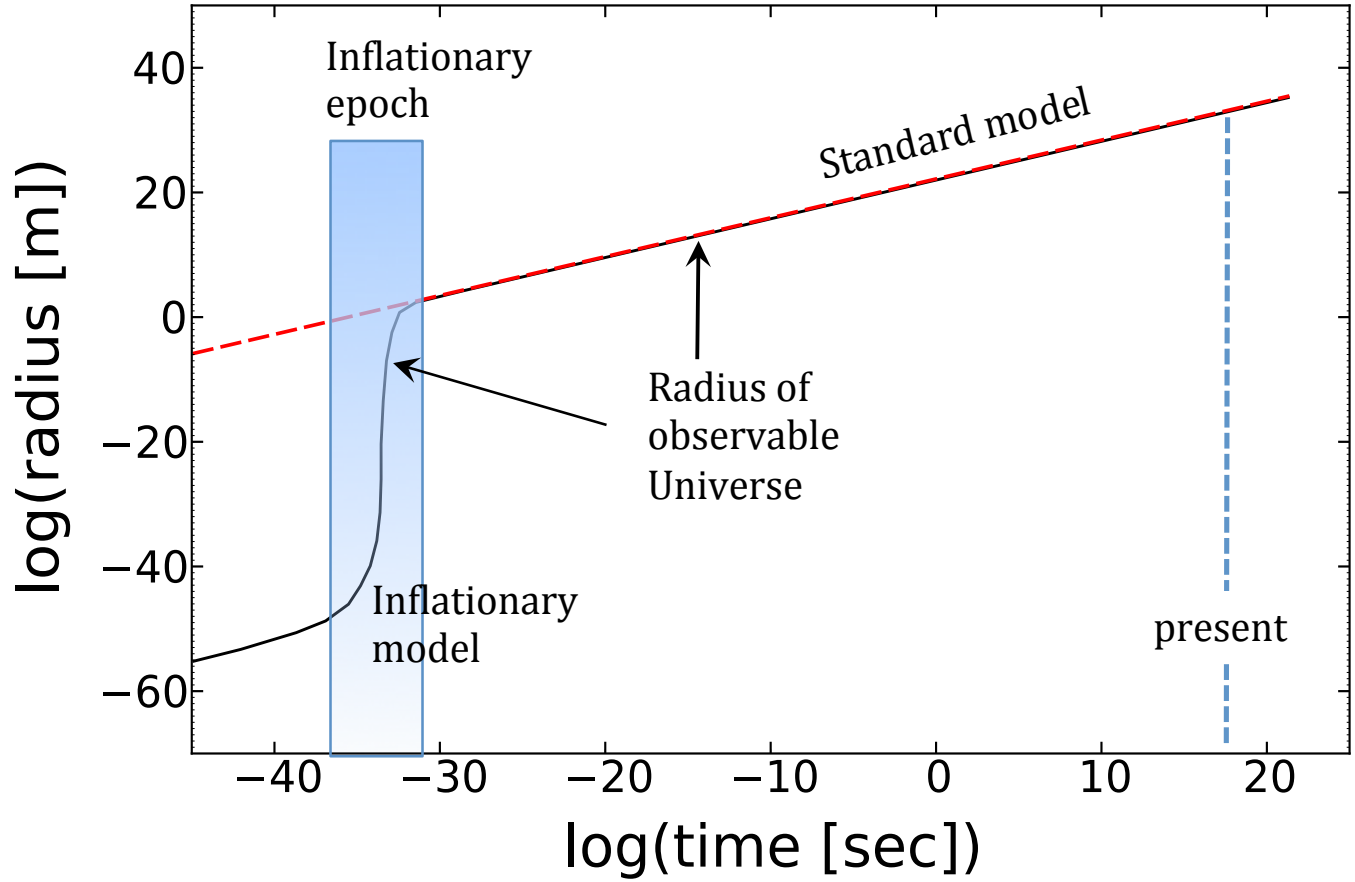
Grand Unification Theory (GUT) of forces

The separation of forces happens in phase transitions with spontaneous symmetry breaking.



kT (temperature)	united forces	big bang time
~ 10^2 GeV	electromagnetic + weak	
~ 10^{13} GeV	electromagnetic + weak + strong	~ 10^{-35} s
~ 10^{19} GeV	electromagnetic + weak + strong + gravity	~ 10^{-43} s Planck time

Inflation solves the flatness problem



In the cosmological models the k/a^2 term, with $k = \pm 1$, could have been important. However, with the sudden inflation the scale factor has increased by ~ 10^{29} and the value of k/a^2 has been reduced by 10^{58}. So, even if we started from a large curvature term there is no other mechanism necessary to arrive at the nearly flat Universe.

During inflation the energy density of the Universe is constant, whereas the scale factor increases exponentially, as described by Eqs. (2.37) to (2.39). This means that Ω must have been exponentially close to unity, $\Omega = 1$ to an accuracy of many decimal places.

Practice 2

1 – Prove Eqs. (2.32) and (2.33) for the closed Universe (**Hint:** introduce the variable change $d\eta = dt/a$).

2 - If $\rho = 3 \times 10^{-27}$ kg m^{-3}, what is the radius of curvature a_0 of Einstein's static Universe? How long would it take a photon to circumnavigate such a Universe?

3 – Suppose the energy density of the cosmological constant is equal to the present critical density $\rho_\Lambda = \rho_0 = 5200$ MeV m^{-3}. What is the total energy of the cosmological constant within a sphere 1 AU in radius?
What is the rest energy of the Sun ($E = Mc^2$)?
Comparing these two numbers, do you expect the cosmological constant to have a significant effect on the motion of planets within the Solar System?

4 – Consider Einstein's static Universe, in which the attractive force of the matter density ρ is exactly balanced by the repulsive force of the cosmological constant, $\Lambda = 4\pi G\rho$. Suppose that some of the matter is converted into radiation (by stars, for instance). Will the Universe start to expand or contract? Explain your answer.