Lecture 3

The quest for the origin of the elements
Cross section $\sigma$

bombard target nuclei with projectiles:

Definition of cross section:

\[
\text{# of reactions per second and target nucleus} = \sigma \times \text{# of incoming projectiles per second and cm}^2
\]

or in symbols: $\text{rate} = \sigma j$

with $j$ as particle number current density.

Units for cross section:

1 barn = $10^{-24}$ cm$^2$ ( = 100 fm$^2$ or about half the size (cross sectional area) of a uranium nucleus)
Mass fraction and abundance

Chemical composition

How can we describe the relative abundances of elements (nuclei) of different species and their evolution in a given sample (say, a star, or the Universe)?

Number density

We could use the number density \( n_i = \text{number of nuclei of species } i \text{ per cm}^3 \)

Disadvantage: tracks not only nuclear processes that create or destroy nuclei, but also density changes, for example due to compression or expansion of the material.

Mass fraction

\( X_i \) is fraction of total mass of sample that is made up by nucleus of species \( i \)

\[
(3.1) \quad n_i = \frac{X_i \rho}{m_i}
\]

\( \rho \): mass density (g/cm\(^3\))

\( m_i \): mass of nucleus of species \( i \)

with

\[
(3.2) \quad m_i \approx A_i \cdot m_u
\]

and

\[
(3.3) \quad m_u = m_{12C}/12 = 1/ N_A
\]

(CGS only !!!) as atomic mass unit (AMU)
Mass fraction and abundance

\[ n_i = \left( \frac{X_i}{A_i} \right) \rho N_A \quad (3.4) \]

call this abundance \( Y_i \)

so

\[ n_i = Y_i \rho N_A \quad (3.5) \]

with

\[ Y_i = \frac{X_i}{A_i} \quad (3.6) \]

\textbf{note:} if the mass is expressed in the CGS unit (grams) then \( 1 \text{[g]} = N_A \cdot m_u \)

(we neglect the binding energies and the mass of the electrons in the atoms)

\[ \sum_i X_i = 1 \quad (3.7) \]

\[ \sum_i Y_i < 1 \quad (3.8) \]

\textbf{The abundance} \( Y \) is proportional to number density but changes only if the nuclear species gets destroyed or produced. Changes in density are factored out.

\textbf{note:} Abundance has no units only valid in CGS

\textbf{of course}

\[ \sum_i X_i = 1 \]

\textbf{but, since} \( Y_i = X_i/A_i < X_i \)
Mass fraction and abundance

Mean molecular weight $\mu_i$

= average mass number =

$$\mu_i = \frac{1}{\sum_i Y_i}$$  \hspace{1cm} (3.10)

Electron Abundance $Y_e$

As matter is electrically neutral, for each nucleus with charge number $Z$ there are $Z$ electrons:

$$Y_e = \sum_i Z_i Y_i$$  \hspace{1cm} (3.11)

We can also write:

$$Y_e = \frac{\sum_i Z_i Y_i}{\sum_i A_i Y_i}$$  \hspace{1cm} (3.13)

So $Y_e$ is the ratio of protons to nucleons in sample (counting all protons including the ones contained in nuclei - not just free protons as described by the “proton abundance”)

$$\langle A \rangle = \frac{\sum_i A_i n_i}{\sum_i n_i} = \frac{\sum_i A_i Y_i}{\sum_i Y_i} = \frac{1}{\sum_i Y_i}$$ \hspace{1cm} (3.9)

or

For 100% hydrogen: $Y_e = 1$

For equal number of protons and neutrons ($N = Z$ nuclei): $Y_e = 0.5$

For pure neutron gas: $Y_e = 0$

$$n_e = \rho N_A Y_e$$  \hspace{1cm} (3.12)

prop. to number of protons

prop. to number of nucleons
Mass fraction and abundance

The plasma within a star is a mix of fully ionized projectiles and target nuclei at a temperature $T$. Using the definition of cross section, we get that for particles with a given relative velocity $v$ and within a volume $V$ with projectile number density $n_i$, and target number density $n_j$:

\[
\lambda = \sigma n_i v
\]
\[
R = \sigma n_i v n_j V
\]

so for reaction rate per second and cm$^3$ is

\[
r = n_i n_j \sigma v
\]

This is proportional to the number of $i$-$j$ pairs in the volume. However, when the species are identical, i.i., $i = j$, one has to divide by 2 to avoid double counting

\[
r = \frac{1}{1 + \delta_{ij}} n_i n_j \sigma v
\]
Reaction rates at temperature $T$

Assuming Local Thermal Equilibrium (LTE), the velocity distribution of particles in a plasma follow a [Maxwell-Boltzmann](https://en.wikipedia.org/wiki/Maxwell-Boltzmann_distribution) velocity distribution:

$$
\Phi(v) = 4\pi \left( \frac{\mu}{2\pi kT} \right)^{3/2} v^2 e^{-\frac{mv^2}{2kT}}
$$

with

$$
\mu = \frac{m_i m_j}{m_i + m_j}
$$

where $\Phi(v)$ is the probability to find a particle with a velocity between $v$ and $v + dv$.

![Maxwell-Boltzmann distribution diagram](https://example.com/maxwell-boltzmann.png)
Reaction rates at temperature $T$

In terms of the Maxwell-Boltzmann distribution, with the mass $m$ replaced by the reduced mass $\mu$ of the $i$ and $j$ particles, i.e.,

$$\mu = \frac{m_i m_j}{m_i + m_j} \quad (3.19)$$

$$\Phi(v) = 4\pi \left( \frac{\mu}{2\pi kT} \right)^{3/2} v^2 e^{-\frac{\mu v^2}{2kT}} \quad (3.20)$$

In the stellar environments, the reaction rates given in Eq. (3.16) has to be averaged over velocities,

$$r = \frac{1}{1 + \delta_{ij} n_i n_j <\sigma v>} \quad (3.21)$$

using over the distribution $\Phi(v)$,

$$r = \frac{1}{1 + \delta_{ij} n_i n_j \int \sigma(v)\Phi(v)v \, dv} \quad (3.22)$$
**Reaction rates in terms of abundances**

In terms of units of stellar reaction rate given by $N_A \langle \sigma v \rangle$ in cm$^3$/s/g, the reaction rate per second and per cm$^3$, is

$$r = \frac{1}{1 + \delta_{ij}} Y_i Y_j \rho^2 N_A^2 \langle \sigma v \rangle \quad (3.23)$$

Sometimes, one uses another notation for reactions per second and target nucleus $j$, by dividing the above equation by $Y_j \rho N_A$, i.e.,

$$\lambda = \frac{1}{1 + \delta_{ij}} Y_i \rho N_A \langle \sigma v \rangle \quad (3.24)$$

One usually call this by the **stellar reaction rate** of a specific reaction.

As an application, let's assume that the species $i$ and $j$ are destructed by $j$ capturing the projectile $i$ in the form $i + j \rightarrow k$. Then

$$\frac{dn_j}{dt} = -n_j \lambda = -n_j Y_i \rho N_A \langle \sigma v \rangle \quad (3.25)$$

$$\frac{dn_k}{dt} = n_k \lambda$$
Since the $\frac{dn_j}{dt}$ is proportional to $-n_j$, the solution of the equation is a decaying exponential. And since the species $k$ is built from the destruction of $j$, its time dependence should be proportional to $[1 - n_j(t)]$. In other words,

$$n_j(t) = n_{0j}e^{-\lambda t}$$  \hspace{1cm} (3.26)

$$n_k(t) = n_{0k}(1 - e^{-\lambda t})$$

In terms of the abundances:

$$Y_j(t) = Y_{0j}e^{-\lambda t}$$  \hspace{1cm} (3.27)

$$Y_k(t) = Y_{0k}(1 - e^{-\lambda t})$$

The lifetime for the destruction of $j$ is

$$\tau = \frac{1}{\lambda} = \frac{1}{Y_j\rho N_A <\sigma v> }$$  \hspace{1cm} (3.28)

After a time equal to a few multiples of $\tau$, the species $j$ is almost completely destroyed while $k$ is almost completely built up from $i$ and $j$ capture. One often uses the name “half-life”, $\tau_{1/2}$, instead of lifetime, where $\tau_{1/2} = \tau \ln 2$. 

Energy generation

Let's consider the energy generation by nuclear reaction in stars. As we know the energy released is given by the Q-values

\[ Q = c^2 \left( \sum_{\text{initial nuclei } i} m_i - \sum_{\text{final nuclei } j} m_j \right) \]  

(3.29)

where \( m \) are the respective masses of the particles. The energy generated per gram and second by a reaction \( i + j \) is then

\[ \varepsilon = \frac{rQ}{\rho} = Q \frac{1}{1 + \delta_{ij}} Y_i Y_j \rho N_A^2 < \sigma v > \]  

(3.30)

We can now define the reaction flow as the abundance of nuclei converted in time \( T \) from species \( j \) to \( k \) via a specific reaction as

\[ F = \int_0^T \left( \frac{dY_j}{dt} \right) \text{ via specific reaction} = \int_0^T \lambda(t) Y_j(t) dt \]  

(3.31)
Branching

Consider possibilities as in the figure on the side in terms of several possible reaction rates $\lambda_i$

The total destruction rate for the nucleus is

$$\lambda = \sum_i \lambda_i$$  \hspace{1cm} (3.32)

And its total lifetime is given by

$$\tau = \frac{1}{\lambda} = \frac{1}{\sum_i \lambda_i}$$  \hspace{1cm} (3.33)

The reaction flow branching into reaction $i$, $b_i$ is the fraction of destructive flow through reaction $i$. (or the fraction of nuclei destroyed via reaction $i$)

$$b_i = \frac{\lambda_i}{\sum_j \lambda_j}$$  \hspace{1cm} (3.34)
The lifetime corresponds to a width in the excitation energy of the state according to Heisenberg

\[ \Delta E \cdot \Delta t = \hbar \]  \hspace{1cm} (3.35)

Therefore, a lifetime \( \tau = \Delta t \) corresponds to a width \( \Gamma = E \):

\[ \Gamma = \frac{\hbar}{\tau} \]  \hspace{1cm} (3.36)

The lifetime for the individual decay modes, or decay “channels” for \( \gamma, p, n, \) and \( \alpha \) emission are usually given as partial widths

\[ \Gamma_{\gamma}, \Gamma_{p}, \Gamma_{n}, \text{ and } \Gamma_{\alpha} \]

with

\[ \Gamma = \sum \Gamma_{i} \]  \hspace{1cm} (3.37)
The nucleons in the nucleus are confined and can only have discrete energies. Therefore, the nucleus, like an atom, can be excited into discrete energy levels, also called by excited states. The figure below illustrates a generic situation.

Nuclear states depend on:
- energy (mass)
- spin
- parity
- lifetimes of $\gamma$, p, n, and $\alpha$ emission
Reaction Types

1. **Fusion & Radiative capture** reactions
   \[ A + C \rightarrow B + \gamma \]

2. **Elastic** collision: entrance channel=exit channel
   \[ A + B \rightarrow A + B : \quad Q = 0 \]

3. **Inelastic** collision (\(Q \neq 0\))
   \[ A + B \rightarrow A^* + B \quad (A^* = \text{excited state}) \]
   \[ A + B \rightarrow A + B^* , \quad \text{etc..} \]

4. **Breakup** reactions
   \[ A(= a + b) + C \rightarrow a + b + C \]

5. **Transfer** reactions
   \[ A + B \rightarrow C + D \]
Reaction Types

• **Transfer cross sections** *(strong interaction)*
  - Non resonant \(^6\text{Li}(p,\alpha)^3\text{He} \)
  - Resonant \(^3\text{He}(d,p)\alpha\)
  - Multiresonance \(^{22}\text{Ne}(\alpha,n)^{25}\text{Mg} \)

• **Capture cross sections** *(electromag interact)*
  - Non resonant \(^6\text{Li}(p,\gamma)^7\text{Be} \)
  - Resonant \(^{12}\text{C}(p,\gamma)^{13}\text{N} \)
  - Multiresonance \(^{22}\text{Ne}(\alpha,\gamma)^{26}\text{Mg} \)

• **Weak capture cross sections** *(weak interaction)*
  - Non resonant \(p(p,e^+\nu)^2\text{H} \)
  - \(^3\text{He}(p,e^+\nu)^4\)
Ex: Radiative capture

Neutron capture $A + n \rightarrow B + \gamma$

I. Direct capture

Direct transition into bound states

II. Resonant capture

Step 1: capture to an unbound state

Step 2: decay to a bound state
Ex: Radiative capture

For radiative capture reactions of the type \( a + A \rightarrow B + \gamma \), QM tell us that we need to know the wavefunctions of the relative motion of the initial set of particles, i.e., \( a + A \), and the wavefunctions of the final particle, i.e., the nucleus \( B \). The energy field of the photon, \( V_\gamma \), is the responsible for the capture process. The equation obtained for the cross section is

\[
\sigma \propto \pi \lambda^2 \left| \langle B \mid V_\gamma \mid a + A \rangle \right|^2 P_1(E)
\]

(3.38)

where \( \pi \lambda^2 \) is a geometrical factor (deBroglie wave length of projectile - “size” of projectile).

\[
\langle B \mid V_\gamma \mid a + A \rangle = \int \Psi^*_B(\mathbf{r}) V_\gamma(\mathbf{r}) \Psi_{a+A}(\mathbf{r}) d^3r
\]

(3.39)

is the interaction matrix element, and \( P_1(E) \) is the penetrability factor, i.e., the probability that the projectile reaches the target nucleus and interacts with it. It depends on the projectile angular momentum \( l \) and its kinetic energy \( E \).

Using

\[
\lambda = \frac{h}{p} = \frac{\hbar}{\sqrt{2mE}}
\]

(3.40)

one gets

\[
\sigma \propto \frac{1}{E} \left| \langle f \mid H \mid a + A \rangle \right|^2 P_1(E)
\]

(3.41)
Angular momentum

Incident particles can have orbital angular momentum $L$. Classically it is given by $L = pb$, where $p$ is the momentum and $b$ the impact parameter.

In quantum mechanics the angular momentum is quantized and can only have discrete values

$$L = \sqrt{l(l + 1)} \hbar$$

(3.42)

Where $l = 0$ s-wave

$l = 1$ p-wave

$l = 2$ d-wave

...

And the parity of the relative motion wave function is given by $(-1)^l$

The particle kinetic energies in stars are pretty low. Let us take $E \sim 10$ keV for a proton, then its momentum is about $p/\hbar = (2mE)^{1/2}/\hbar \sim 0.02$ fm$^{-1}$. The protons can only react with other protons if they come close to each within a distance of $b \sim 1$ fm. The angular momentum under these conditions is thus $l \sim pb/\hbar \sim 0.02 << 1$.

Thus, at astrophysical energies only very low values of $l$ (mostly s-waves) are of relevance.
**Transmission of penetrability: neutrons**

For a neutron incident on a nucleus, there is not Coulomb repulsion and the potential looks like in the figure below

![Potential diagram](image)

The change in potential from a free neutron outside the nucleus to the nuclear attraction as it penetrates the nucleus still causes a quantum mechanical reflection. So, the penetrability is not constant (or just one), but depends on the energy as

\[ P \sim \sqrt{E} \]  
(3.43)

The cross section is thus

\[ \sigma \propto \frac{1}{\sqrt{E}} \]  
(3.44)  

or

\[ \sigma \propto \frac{1}{V} \]  
(3.45)
Example:

\[ ^7\text{Li}(n,\gamma) \sim \frac{1}{v} \]

Deviation from \( \frac{1}{v} \) due to resonant contribution

Because \( \sigma \propto \frac{1}{v} \), then \( \sigma v = \text{const} = \langle \sigma v \rangle \) \hspace{1cm} (3.46)
Penetrability for charged particles

When two nuclei with charge $Z_1 e$ and $Z_2 e$ approach each other they repel due to the Coulomb force. When they come within the range of nuclear forces the repulsion turns into a strong attraction. In terms of the potential energy $V$ (recall that force $F = -\frac{dV}{dr}$).

At the top of the Coulomb barrier the potential is

$$V_c = \frac{Z_1 Z_2 e^2}{R}$$

(3.47)

or

$$V_c [\text{MeV}] = 1.44 \frac{Z_1 Z_2}{R [\text{fm}]} \approx 1.2 \frac{Z_1 Z_2}{(A_1^{1/3} + A_2^{1/3})}$$

(3.48)

For example, if we consider the reaction $^{12}\text{C}(p,\gamma)$ (relevant in the CN cycle), then $V_c = 3 \text{ MeV}$. In start the typical particle energies when this reaction occurs is $kT = 10 \text{ keV}$. This is much smaller than the Coulomb barrier, meaning that the particle has to tunnel (a pure QM effect) to get close together and fuse together.
Each transmission probability $T_i$ through each barrier can be obtained easily. This way of calculating the total transmission probability is called the **WKB approximation**.

$$ T_i(E) = \exp \left[ -\frac{2}{\hbar} \Delta \sqrt{2\mu (V - E)} \right] $$

(3.49)

$$ T(E) = \prod_i T_i(E) $$

(3.50)

$$ T(E) = \exp \left[ -\frac{2}{\hbar} \int_0^D \sqrt{2\mu} \left[ V(r) - E \right] dr \right] $$

(3.51)
Tunneling through a Coulomb barrier

For a pure Coulomb barrier, this leads to

\[ P_l(E) \propto e^{-2\pi\eta} \quad (3.52) \]

where

\[ \eta = \sqrt{\frac{\mu}{2E}} \frac{Z_1 Z_2 e^2}{\hbar} \quad (3.53) \]

This implies that

\[ \sigma \propto \frac{1}{E} \left| \langle f | H | a + A \rangle \right|^2 P_l(E) \sim \frac{1}{E} e^{-2\pi\eta(E)} S(E) \quad (3.54) \]

Where \( S(E) \) contains the information of what happens when the particles are inside the barriers, i.e., when they interact via the strong force.
Example: $^{13}\text{C}(p,\gamma)$

The penetrability $P_l(E)$ for charged particles decrease very fast as a function of $E$. And so does the cross section. An example is shown in the figure for the $^{12}\text{C}(p,\gamma)$ cross section. The dots are experimental data.

Very small cross sections cannot be measured because the number of events (reactions) occurring per unit time is very small requiring a very long time for data taking (expensive, or even impossible).

We need the cross section here!
Astrophysical $S$-factor

Coulomb barrier for charged particles

Cross section has a steep energy dependence

\[ \sigma(E) = \frac{1}{E} S(E) \exp \left[ -2\pi \frac{Z_1 Z_2 e^2}{\hbar v} \right] \] (3.55)

Extrapolation of data to low astrophysics energies better done with astrophysical $S$-factor $S(E)$

The $S$-factor can be:
- easier graphed
- easier fitted and tabulated
- easier extrapolated
- and contains all the essential nuclear physics
Astrophysical $S$-factor for the reaction $^4\text{He} + ^3\text{He} \rightarrow ^7\text{Li} + \gamma$. 

![Graph showing the astrophysical S-factor for the reaction $^4\text{He} + ^3\text{He} \rightarrow ^7\text{Li} + \gamma$. The graph plots $S_{34}(E)$ (keV b) against energy $E$ (MeV). Data points from Seattle, Weizmann, ERNA, and LUNA are shown with error bars.]
Note that the relevant cross section is in tail of MB distribution, which is much larger than $kT$. This is very different than n-capture which occurs at energies $\sim kT$. Nonetheless, one has always $E_0 \ll E_{coul}$ (Coulomb barrier) and only a small number of partial waves ($l = 0$ mainly) contribute.
Reaction rate estimates

By approximating the Gamow window by a Gaussian function centered at the energy $E_0$, one can calculate the reaction rate in an approximate form

\[
<\sigma v> = \frac{16}{9\sqrt{3}} \frac{1}{\mu} \frac{1}{2\pi \alpha Z_1 Z_2} S(E_0) \left( \frac{3E_0}{kT} \right)^2 \tag{3.56}
\]

where

\[
E_0 = \left( \frac{b k T}{2} \right)^{3/2} = 0.12204 \left( Z_1^2 Z_2^2 A \right)^{1/3} T_9^{2/3} \text{ MeV} \tag{3.57}
\]

\[
N_A <\sigma v> = 7.83 \cdot 10^9 \left( \frac{Z_1 Z_2}{A T_9^2} \right)^{1/3} S(E_0) [\text{MeV barn}] e^{-4.2487 \left( \frac{Z_1^2 Z_2^2 A}{T_9} \right)^{1/3}} \tag{3.58}
\]

in cm\(^3\)/(s.mole). One still needs to know (measure, calculate) the cross section, or $S$-factor, at the Gamow energy $E_0$. 

The quest for the origins of the elements, C. Bertulani, RISP, 2016
Reaction rates - examples

1 MeV $\sim 10^{10}$ K

$T_9 = 10^9$ K

$E_0 \sim 0.122 \left( \frac{Z_i^2 Z_j^2 A}{A_i + A_j} \right)^{1/3} T_9^{2/3}$ MeV

\[ A = \frac{A_i A_j}{A_i + A_j} \] (3.59)

<table>
<thead>
<tr>
<th>Reaction</th>
<th>$T$ ($10^9$ K)</th>
<th>$E_0$ (MeV)</th>
<th>$E_{\text{coul}}$ (MeV)</th>
<th>$\sigma(E_0)/\sigma(E_{\text{coul}})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d + p$</td>
<td>0.015</td>
<td>0.006</td>
<td>0.3</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>$^3\text{He} + ^3\text{He}$</td>
<td>0.015</td>
<td>0.021</td>
<td>1.2</td>
<td>$10^{-13}$</td>
</tr>
<tr>
<td>$\alpha + ^{12}\text{C}$</td>
<td>0.2</td>
<td>0.3</td>
<td>3</td>
<td>$10^{-11}$</td>
</tr>
<tr>
<td>$^{12}\text{C} + ^{12}\text{C}$</td>
<td>1</td>
<td>2.4</td>
<td>7</td>
<td>$10^{-10}$</td>
</tr>
</tbody>
</table>

\[
\left\langle \sigma v \right\rangle \sim \int \sigma(E) E \exp \left[ -\frac{E}{kT} \right] dE
\] (3.60)

\[
E_{\text{Coul}} \sim 1.2 \frac{Z_i Z_j}{A_i^{1/3} + A_j^{1/3}} \text{ MeV}
\] (3.61)
Resonant reactions

In some reactions, the energy of the formed nucleus B might coincide with a quantum state that would be in the system if the nuclear confining well would be infinitely high (impenetrable walls). These are like excited states above the particle a separation energy from B, which we call $S_a$. Such states serve as resonances.
Resonant reactions

If the reaction occurs close to a resonance, the cross section contribution due to the resonance is given by the Breit-Wigner formula

\[ \sigma(E) = \pi \hat{\lambda}^2 \omega \frac{\Gamma_a \Gamma_b}{(E - E_r)^2 + (\Gamma/2)^2} \]  

(3.62)

where \( \omega \) is a statistical factor given by

\[ \omega = \frac{2J_r + 1}{(2J_1 + 1)(2J_2 + 1)} \]  

(3.63)

\( \Gamma_a \) is the partial width for decay of resonance by emission of particle \( a = \) rate for formation of \( B \) nucleus. \( \Gamma_b \) is the partial width for decay of resonance by emission of particle \( b \).

\( \Gamma = \Gamma_a + \Gamma_b \) is the total width is in the denominator as a large total width reduces the relative probabilities for formation and decay into specific channels.
Assuming a narrow resonance, then one can carry out the integral for the reaction rate can be done analytically and one finds, using

$$
\int_0^\infty dE \frac{\Gamma_1 \Gamma_2}{(E - E_r)^2 + (\Gamma/2)^2} = \frac{\pi \Gamma_1 \Gamma_2}{\Gamma}, \quad \text{for } \Gamma \ll E_r
$$  \hspace{1cm} (3.64)

that the contribution of a single narrow resonance to the stellar reaction rate is given by

$$
N_A < \sigma v > = 1.54 \cdot 10^{11} (A T_9)^{-3/2} \omega \gamma [\text{MeV}] e^{-11.605 E_r [\text{MeV}] T_9} \text{ cm}^3 \text{ s mole}^{-1}
$$  \hspace{1cm} (3.65)

The rate is entirely determined by the “resonance strength”

$$
\omega \gamma = \frac{2 J_r + 1}{(2 J_1 + 1)(2 J_T + 1)} \frac{\Gamma_1 \Gamma_2}{\Gamma}
$$  \hspace{1cm} (3.66)

and this has to be determined experimentally for the angular momenta and partial widths for the decay of nucleus B by emission of particles a and b.
Stellar Modeling: nuclear reaction networks

Mass fraction of nuclear species $X$
Abundance $Y = X/A$ (A=mass number)
Number density $n = \rho N_A Y$ ($\rho$=mass density, $N_A$=Avogadro)

$\frac{dY_i}{dt} = \sum_j N_{j}^{i} \lambda_{j} Y_{j} + \sum_{jk} N_{jk}^{i} \rho N_A \langle \sigma v \rangle Y_{j} Y_{k} + ...$ (3.67)

Nuclear energy generation

Observations

$N_i^{..}$: number of nuclei of species I produced (positive) or destroyed (negative) per reaction
In the stellar environments, the reaction rates have to be averaged over velocities with the distribution $\Phi(v)$,

$$r = \frac{1}{1 + \delta_{ij}} n_i n_j < \sigma v >$$

$$r = \frac{1}{1 + \delta_{ij}} n_i n_j \int \sigma(v) \Phi(v) v \, dv$$
Complications in stellar environment

Beyond certain temperature and density, there are additional effects related to the extreme stellar environments that affect reaction rates.

In particular, experimental laboratory reaction rates need a (theoretical) correction to obtain the stellar reaction rates.

The most important two effects are:

1. Thermally excited target

At high stellar temperatures photons can be absorbed and excite the target. Reactions on excited target nuclei can have different angular momentum and parity selection rules and have a somewhat different Q-value. In the laboratory it is nearly impossible to prepare targets under such conditions.

2. Electron screening

In a stellar environment, atoms are fully ionized, but the electron gas still shields the nucleus and affects the effective Coulomb barrier.

In the laboratory, the reactions are also screened by the atomic electrons, but the screening effect is by bound electrons.
Electron screening (in stars)

In stars the particles are surrounded by free electrons. Once can use the Debye-Hueckel screening (Salpeter 1959). This approximation works only if

\[
n R_D^3 \gg 1
\]  
(3.65)

One then finds a correction of the form

\[
f(E) = e^{U_e/kT}
\]

\[
R_D \sim \sqrt{kT / \rho} \sim 0.218 \text{ Å (Sun)}
\]  
(3.70)

\[
\langle \sigma v \rangle_{\text{plasma}} = f(E) \langle \sigma v \rangle_{\text{bare}}
\]

\[
V_{\text{eff}} = \frac{Ze^2}{r} e^{-r/R_D}
\]

\[7\text{Be}(p, \gamma)\^8\text{B} \ (T \sim 10^7 \text{K}) : \quad f(E) \sim 1.2 \]  
(20 % effect)  
(3.71)
Electron screening (in stars)

But there are problems: Gamow energy >> thermal energy (dynamical screening) → Debye sphere is not really spherical

\[ \alpha = \frac{v}{v_{\text{thermal}}} , \quad s_{12} = \text{dynamic} / \text{Debye} \]

<table>
<thead>
<tr>
<th>Reaction</th>
<th>Gamow Energy ((E_g/T))</th>
<th>(s_{12})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p-p)</td>
<td>4.6</td>
<td>0.76</td>
</tr>
<tr>
<td>(^3\text{He} - ^3\text{He})</td>
<td>16.6</td>
<td>0.75</td>
</tr>
<tr>
<td>(^3\text{He} - ^4\text{He})</td>
<td>17.3</td>
<td>0.76</td>
</tr>
<tr>
<td>(p-^7\text{Be})</td>
<td>13.9</td>
<td>0.80</td>
</tr>
<tr>
<td>(p-^{14}\text{N})</td>
<td>20.6</td>
<td>0.82</td>
</tr>
</tbody>
</table>


Also, the number of particles within the Debye sphere is not so large to justify the mean-field approximation → Fluctuations in the Debye sphere should be important.

- One cannot derive the screening from thermodynamics but one has to resort to kinetic equations.
- Deviations from Debye-Hueckel can be large.
In the adiabatic model one assumes that the projectile acquires atomic binding energy as it penetrates the electron cloud of the target. The extra amount of energy is due to the different atomic bindings for A and B which can be taken from an atomic table. One then finds the extra $\Delta E$

$$\Delta E = E' - E$$  \hspace{1cm} (3.72)

With this one can calculate the change in the cross section:

$$\sigma_{\text{fusion}}^{\text{lab}} \sim \sigma_{\text{bare}}(E + \Delta E)$$

$$\sim \exp \left[ \pi \eta(E) \frac{\Delta E}{E} \right] \sigma_{\text{bare}}(E)$$  \hspace{1cm} (3.73)
Electron screening (in the laboratory)

But the amount of screening needed to explain the experimental data is more than either given by the adiabatic model, or by a more complex theoretical calculation (dynamic screening).

<table>
<thead>
<tr>
<th>Rolfs, 1995</th>
<th>$\Delta E$ [eV] experiment</th>
<th>$\Delta E$ [eV] adiabatic limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reaction</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$d(^3\text{He}, p)^4\text{He}$</td>
<td>180 ± 30</td>
<td>119</td>
</tr>
<tr>
<td>$^6\text{Li}(p, \alpha)^3\text{He}$</td>
<td>470 ± 150</td>
<td>186</td>
</tr>
<tr>
<td>$^6\text{Li}(d, \alpha)^4\text{He}$</td>
<td>380 ± 250</td>
<td>186</td>
</tr>
<tr>
<td>$^7\text{Li}(p, \alpha)^4\text{He}$</td>
<td>300 ± 280</td>
<td>186</td>
</tr>
<tr>
<td>$^{11}\text{B}(p, \alpha)^4\text{He}$</td>
<td>620 ± 65</td>
<td>348</td>
</tr>
</tbody>
</table>

Also, very enhanced electron screening has been found in solids, and liquids.

*Czerski, Kasagi, 2010*
1 - (a) What is the most probable kinetic energy of a hydrogen atom at the interior of the Sun (T = 1.5 \times 10^7 \text{ K})? (b) What fraction of these particles would have kinetic energy in excess of 100 keV? (you need to perform an integral here).

2 - Calculate the Gamow energy $E_0$ for the $^{14}\text{N}(p,\gamma)^{15}\text{O}$ reaction for temperatures of 0.01 to 0.1 \text{G K}.

3 - Show that the MB distribution, Eq. (3.20), can be deduced from the Boltzmann distribution with $E = mv^2/2$ and the normalization condition,

$$\int \Phi(v) \, dv = 1 \quad (3.74)$$

4 - What is the penetrability factor for the reaction of problem 2 at the two Gamov energies.

5 - Calculate the reaction rate for $^{14}\text{N}(p,\gamma)^{15}\text{O}$ for the resonant contribution, with $E = 0.259 \text{ MeV}, (\omega \gamma)_r = 13 \text{ meV}$ (see Eq. (3.65)) and for the non-resonant component, with $S = 1.61 \text{ keV b}$ (see Eq. (3.58)).