

# Nuclear astrophysics studies with ultra-peripheral heavy-ion collisions

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**Abstract.** I describe in very simple terms the theoretical tools needed to investigate ultra-peripheral nuclear reactions for nuclear astrophysics purposes. For a more detailed account, see ref. [1].

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**Semiclassical coupled-channels equations.** Consider  $H = H_0 + V$ , where  $H$  is the Hamiltonian composed by a non-perturbed Hamiltonian  $H_0$  and a small perturbation  $V$ . The Hamiltonian  $H_0$  satisfies an eigenvalue equation  $H_0\psi_n = E_n\psi_n$ , whose eigenfunctions form a complete basis in which the total wavefunction  $\Psi$ , that obeys the Schrödinger equation  $H\Psi = i\hbar\partial\Psi/\partial t$ , can be expanded:

$$\Psi = \sum_n a_n(t) \psi_n e^{-iE_n t/\hbar}. \quad (1)$$

Inserting this expansion in the Schrödinger equation, one obtains

$$i\hbar \sum_n \dot{a}_n \psi_n e^{-iE_n t/\hbar} = \sum_n V a_n \psi_n e^{-iE_n t/\hbar}, \quad (2)$$

with  $\dot{a}_n \equiv da_n(t)/dt$ . Using the orthogonalization properties of the  $\psi_n$ , we multiply (2) by  $\psi_k^*$  and integrate over the coordinate space to get the *coupled-channels equations*

$$\dot{a}_k(t) = -\frac{i}{\hbar} \sum_n a_n(t) V_{kn}(t) e^{i\frac{E_k - E_n}{\hbar}t}, \quad (3)$$

where the matrix element ( $d\tau$  is the coordinate volume element) is  $V_{kn} = \int \psi_k^* V \psi_n d\tau$ .

Often, the perturbation  $V$  is very small and the system, initially in state  $n=0$ , does not change appreciably. Thus we can insert  $a_n = \delta_{n0}$  in the right-hand side of the equation (3), which yields

$$a_k(t) = \int_{-\infty}^t dt V_{kn}(t) e^{i(E_k - E_0)t/\hbar}. \quad (4)$$

This is the *first-order perturbation theory* result. Obtaining the coefficients  $a_n$ , the probability of occupation of state  $n$  is simply given by  $|a_n(\infty)|^2$ . Isn't that fun? So simple, yet so powerful! In energy space, things are not so different (I will be back to this later).

**Multipole expansions.** Let us now consider a point particle with charge  $Z_p e$  at a distance  $\mathbf{r}$  from the center of a charge distribution where we put the origin of our frame

of reference. At a point  $\mathbf{r}'$  from the origin let the charge density be  $\rho(\mathbf{r}')$ . The potential created by the charge  $Z_p e$  located at  $\mathbf{r}$ , averaged over the charge distribution, is

$$V_C(\mathbf{r}) = Z_p e \int d^3 r' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{Z_p Z_T e^2}{r} + \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^3} + \frac{Q_{ij} r_i r_j}{2r^5} + \dots, \quad (5)$$

where  $\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d^3 r'$  and  $Q_{ij} = \int (3r'_i r'_j - r'^2) \rho(\mathbf{r}') d^3 r'$  are the *dipole* and *quadrupole moments* of the charge distribution, respectively. In the last step we have expanded the factor  $1/|\mathbf{r} - \mathbf{r}'|$  for  $r \ll r'$ , rearranged terms, and used  $\int \rho(\mathbf{r}') d^3 r' = Z_T e$ . No big deal. Try deriving it yourself. I bet you can do it even after eating pasta with chianti.

**Semiclassical what?!** In the semiclassical approximation, one assumes that the projectile's coordinate  $\mathbf{r}$  can be replaced by  $\mathbf{r}(t)$ , following the classical trajectory of the projectile. The dependence on  $\mathbf{r}'$  is used to treat  $\mathbf{p}$  and  $Q_{ij}$  as operators. The excitation of nucleus  $T$  by nucleus  $P$  is obtained by using the matrix element  $\langle f | V_C(\mathbf{r}', t) | i \rangle$ , where  $|i\rangle$  ( $|f\rangle$ ) is initial (final) state of nucleus  $T$ . In layperson's terms, it means the replacement of the static density  $\rho(\mathbf{r}')$  by the *transition density*  $\rho_{fi}(\mathbf{r}') = \Psi_f^*(\mathbf{r}') \Psi_i(\mathbf{r}')$  in Eq. (5), where  $\Psi_i$  ( $\Psi_f$ ) is the initial (final) wavefunction of nucleus  $T$ .

The validity of the semiclassical approximation relies on the smallness of the wavelength  $\lambda = \hbar/2\pi$  of the projectile's motion as compared to the distance of closest approach between the nuclei. Let us consider a central collision. The distance of closest approach is  $a = 2a_0 = Z_p Z_T e^2 / E$ , where  $E = mv^2/2$  is the kinetic energy of relative motion between the projectile and the target and  $m$  is the reduced mass. We can use the so-called *Sommerfeld parameter*,  $\eta = a_0 / \lambda$  to measure the validity of the semiclassical approximation. Since  $\lambda = \hbar/p = \hbar/mv$ , we have  $\eta = Z_T Z_p e^2 / mv \gg 1$  as the condition for validity of the semiclassical approximation. It is easy to verify that for nucleus-nucleus collisions this condition is valid for most cases of interest. In summary, the “*semi*” from semiclassical means “*quantum*”. The “*classical*” means that the scattering part is treated in classical terms. The later is well justified in most situations. But don't worry, we can also do all quantum easily, as I show later.

**Low energy central collisions.** The fun part starts here. Consider the transition of the ground state  $J = 0$  of a deformed nucleus to an excited state with  $J = 2$  as a result of a frontal collision with scattering angle of  $\theta = 180^\circ$ . From Eq. (5) the perturbing potential is  $V = \frac{1}{2} Z_p e^2 Q_{if} / r^3$ . According to Eq. (4), the excitation amplitude to first order is

$$a_{if} = \frac{Z_p e^2 Q_{if}}{2i\hbar} \int \frac{e^{i\omega t}}{r^3} dt. \quad (6)$$

At an scattering of  $\theta = 180^\circ$  a relationship exists between the separation  $r$ , the velocity  $v$ , the initial velocity  $v_0$ , given by  $v = dr/dt = \pm v_0 (1 - a/r)^{1/2}$  (show this using energy conservation). If the excitation energy is small, we can assume that the factor  $e^{i\omega t}$  in (6) does not vary much during the time that the projectile is close to the nucleus. The orbital integral is then solved easily by the substitution  $u = 1 - a/r$ , resulting in (you can do this integral, I know)

$$a_{if} = \frac{4Z_p e^2 Q_{if}}{3i\hbar v_0 a^2} = \frac{4Q_{if} E^2}{3Z_p e^2 \hbar v_0 Z_T^2}. \quad (7)$$

The differential cross section is given by the product of the Rutherford differential cross section at  $180^\circ$  and the excitation probability along the trajectory, measured by the square of  $a_{if}$ . That is,  $d\sigma/d\Omega|_{\theta=180^\circ} = d\sigma_R/d\Omega|_{\theta=180^\circ} \times |a_{if}|^2$ . One obtains

$$\left. \frac{d\sigma}{d\Omega} \right|_{\theta=180^\circ} = \frac{m_0 E |Q_{if}|^2}{18\hbar^2 Z_T^2}, \quad (8)$$

where  $m_0$  is the reduced mass of the projectile+target system. Oops! This expression is independent of the charge of the projectile. But you have always heard that heavy ions (large  $Z_P$ ) are more efficient for Coulomb excitation. What is wrong? Show that there is nothing wrong. This formula is right and what you've heard is correct.

**General multipole expansion.** Instead of Eq. (5), for electric excitations an exact multipole expansion can be carried out with the help of spherical harmonics:

$$V(\mathbf{r}) = \sum_{LM} \frac{4\pi}{2L+1} \frac{1}{r^{L+1}} Y_{LM}^*(\mathbf{r}) \mathcal{M}(EL, M). \quad (9)$$

The *electric multipole moment of rank*  $L = 0, 1, \dots$  is given by  $\mathcal{M}(EL, M) = \int d^3r' \rho(\mathbf{r}') r'^L Y_{LM}(\mathbf{r}')$ , where  $M = -L, -L+1, \dots, +L$ .

In semiclassical calculations,  $\mathbf{r}$  is replaced by a time-dependent coordinate along a Rutherford trajectory for the relative motion,  $\mathbf{r}(t)$ . As before, we can proceed to calculate the excitation amplitude for the transition of the target from a state with energy  $E_i$  to a state with energy  $E_f$  by replacing  $\rho$  by  $\rho_{if}$  (or  $\mathcal{M}$  by  $\mathcal{M}_{if}$ ) and integrating over time from  $-\infty$  to  $+\infty$ . The cross section is obtained by squaring the excitation amplitude, summing it over final and averaging over the initial intrinsic angular momenta of the target. Oh, don't forget to multiply it by the Rutherford trajectory. One gets (with some omitted factors)

$$\frac{d\sigma_{EL}}{d\Omega} \simeq Z_P^2 B(EL) |I_{EL}(\omega_{fi})|^2, \quad (10)$$

where  $\omega_{fi} = (E_f - E_i)/\hbar$  and  $B(EL) \simeq \int dr' r'^L \rho_L(r')$  is the *reduced transition probability*. By the way,  $\rho_L(r')$  is the radial part of the L-pole component of the transition density  $\rho_{if}(\mathbf{r}')$ . The factor  $I_{EL}$  involves a sum over  $M$  of the *orbital integrals*  $I_{ELM}(\omega_{fi})$  given by

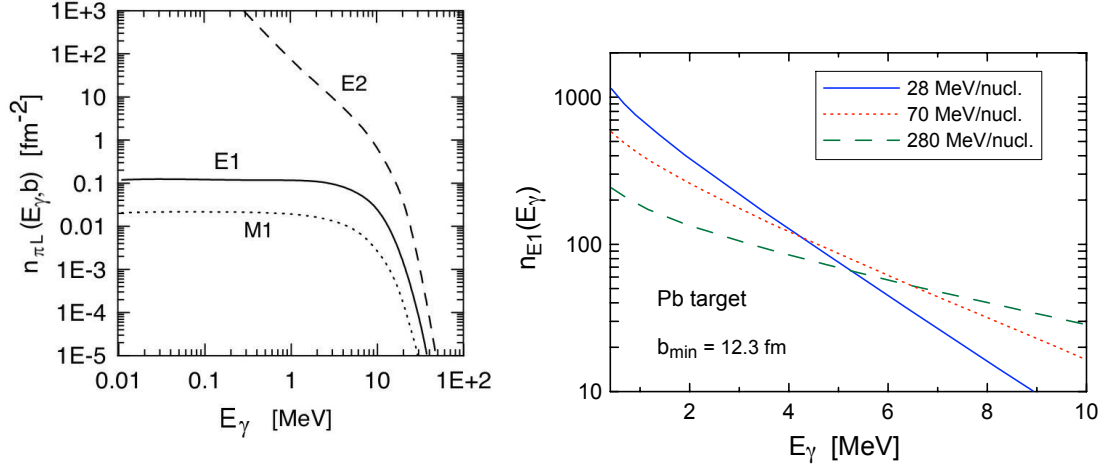
$$I_{ELM}(\omega_{fi}) = \int dt \frac{1}{r^{L+1}(t)} Y_{LM}(\hat{\mathbf{r}}(t)) e^{i\omega_{fi}t}. \quad (11)$$

**Virtual photon numbers.** Integration of (10) over all energy transfers  $E_\gamma = E_f - E_i$ , yields

$$\frac{d\sigma_C}{d\Omega} = \sum_{EL} \frac{d\sigma_{EL}}{d\Omega} = \sum_{EL} \int \frac{dE_\gamma}{E_\gamma} \frac{dn_{EL}}{d\Omega}(E_\gamma) \sigma_\gamma^{EL}(E_\gamma), \quad (12)$$

where  $\sigma_\gamma^{EL}$  are the *photonuclear cross sections*,

$$\sigma_\gamma^{EL}(E_\gamma) \simeq E_\gamma^{2L-1} B(EL). \quad (13)$$



**FIGURE 1.** Left: Equivalent photon numbers per unit area incident on  $^{208}\text{Pb}$ , in a collision with  $^{16}\text{O}$  at 100 MeV/nucleon and with impact parameter  $b = 15$  fm, as a function of the photon energy  $E = \hbar\omega$ . The curves for the E1, E2 and M1 multiplicities are shown. Right: Total number of virtual photons for the E1 multipolarity, “as seen” by a projectile passing by a lead target at impact parameters  $b_{\min} = 12.3$  fm and larger (i.e., integrated over impact parameters), for three typical bombarding energies.

The virtual photon numbers,  $n_{EL}(E_\gamma)$ , are given by

$$\frac{dn_{EL}}{d\Omega}(E_\gamma) \simeq Z_p^2 |I_{EL}(E_\gamma)|^2. \quad (14)$$

The dependence of the cross section on the deflection angle is included in  $dn_{EL}/d\Omega$ . Since for a Rutherford trajectory the deflection angle is related to the impact parameter by  $b = a_0 \cot \theta/2$ , we can write the cross section in terms of an impact parameter dependence by using  $dn_{EL}/2\pi b db \simeq \sin^4(\theta) dn_{EL}/d\Omega$ .

The formalism above can be extended to treat magnetic multipole transitions,  $ML$ . It is much more complicated (involves currents, spins): hard work, but straight-forward.

The reactions induced by real photons include the contribution of all multiplicities with the same weight, i.e.,  $\sigma_\gamma(E_\gamma) = \sum_{E/M,L} \sigma_\gamma^{E/M,L}(E_\gamma)$ , whereas according to Eq. (12), the excitation by virtual photons (e.g. Coulomb excitation) has different weights,  $n_{E/M}$ , for different multiplicities.

Figure (1) (left) shows the equivalent photon numbers per unit area,  $dn_{EL}/2\pi b db$ , incident on  $^{208}\text{Pb}$ , in a collision with  $^{16}\text{O}$  at 100 MeV/nucleon and with impact parameter  $b = 15$  fm, as a function of the photon energy  $E = \hbar\omega$ . The curves for the E1, E2 and M1 multiplicities are shown. One sees that there is a cutoff for excitation energies beyond the adiabatic limit, i.e.,  $E_\gamma < \gamma\hbar v/b$ . On the right we show the total number of virtual photons,  $n_{EL} = \int_{b_{\min}}^\infty db dn_{EL}/(2\pi b db)$ , for the E1 multipolarity, “as seen” by a projectile passing by a lead target at impact parameters  $b_{\min} = 12.3$  fm and larger (i.e., integrated over impact parameters), for three typical bombarding energies. Lesson: more and more high energy photons are available as the beam energy increases.

We can easily understand the origin of the adiabatic condition by investigating the orbital integral, Eq. (11). Notice that for times larger than  $t_{\text{exc}} = 1/\omega$  the integral

oscillates too fast and  $I_{ELM}$  is small. For collisions at low energies, the collision time is given by  $t_{coll} = a_0/v$ , where  $a_0 = Z_P Z_T e^2 / 2E_{c.m.}$ . Thus, the excitation is possible if  $t_{coll}/t_{exc} < 1$  otherwise the system will respond adiabatically (i.e. nothing interesting happens). This condition is called the *adiabatic condition*,  $\omega a_0/v < 1$ . For collisions at high energies, nuclei follow nearly straight-line orbits and it is more appropriate to use the impact parameter,  $b$ , as a measure of the distance of closest approach. The collision time is  $t_{coll} = b/\gamma v$ , where  $\gamma = 1/\sqrt{1-v^2/c^2}$  is the Lorentz contraction factor. Thus the adiabatic condition becomes  $\omega b/\gamma v < 1$ . Assuming an impact parameter of 20 fm, states with energy up to  $\gamma \hbar v / (20 \text{ fm})$  can be appreciably excited. Thus, even for moderate values of  $\gamma$ , i.e.,  $\gamma = 1 - 2$ , it is possible to excite giant resonances. With increasing bombarding energies, ultraperipheral collisions can access the quasi-deuteron effect, produce deltas, mesons (e.g.,  $J/\Psi$ ), even the Higgs boson. Whatever! You name it.

**Nuclear response to multipolarities.** The response function is

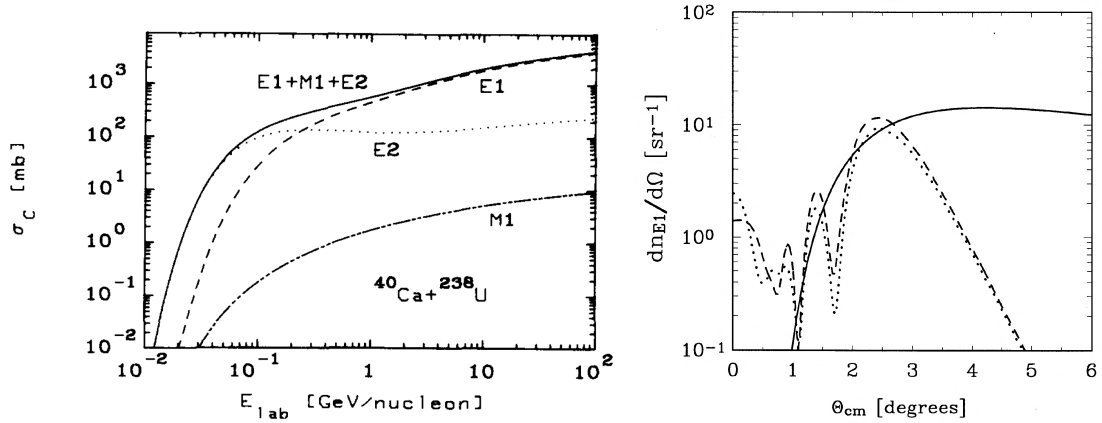
$$B(EL) \sim \left| \int r^L \rho_{if} d^3r \right|^2, \quad (15)$$

where  $\rho_{if} = \Psi_f^* \Psi_i$  is the transition density. A simple estimate can be done for the excitation of high multipolarities by assuming that the wavefunctions have the form  $\Psi_i = \Psi_f = 1/\sqrt{R^3}$ , which yields  $B(EL) \sim R^{2L}$ , or, from Eq. (13),  $\sigma_L^\gamma \sim (kR)^{2L}$ , where  $k = E_\gamma/\hbar c$ . Thus,  $\sigma_{L+1}/\sigma_L \sim (kR)^2$ . Usually  $kR \ll 1$  (long-wavelength approximation) for low-lying states and we see that the cross sections decrease strongly with multipolarity.

It is useful to estimate the total photoabsorption cross section summed over all transitions  $|i\rangle \rightarrow |f\rangle$ . Such estimates are given by *sum rules* (SR) which approximately determine quantities of the following type:  $\mathcal{S}[\mathcal{O}] = \sum_f (E_f - E_i) \left| \langle f | \mathcal{O} | i \rangle \right|^2$ . Here the transition probabilities for an arbitrary operator  $\mathcal{O}$  are weighted with the transition energy. For such *energy-weighted* sum rules (EWSR),  $\mathcal{S}$ , a reasonable estimate can be derived for many operators under certain assumptions about the interactions in the system. Using the completeness of the intermediate states, the commutation relations between the Hamiltonian and the operator  $\mathcal{O}$ , assuming that the Hamiltonian does not contain momentum-dependent interactions, one gets the EWSR for the dipole operator,  $d_z = r Y_{10}(\hat{\mathbf{r}})$ ,  $\mathcal{S}[d_z] = \sum_a \hbar^2 e_a^2 / 2m_a$ , where the sum extends over all particles with mass  $m_a$  and charge  $e_a$ . This is the old *Thomas-Reiche-Kuhn* (TRK) dipole SR.

We have to exclude the center-of-mass motion. Therefore our  $z$ -coordinates should be intrinsic coordinates,  $z_a \Rightarrow z_a - R_z$ , where  $R_z = \sum_a z_a / A$ . Hence, the intrinsic dipole moment is  $d_z = \sum_a e_a (z_a - R_z) = e \sum_p z_p - (Ze/A) (\sum_p z_p + \sum_n z_n)$ . This operator can be rewritten as  $d_z = e_p \sum_p z_p + e_n \sum_n z_n$  where protons and neutrons carry *effective charges*  $e_p = eN/A$ ,  $e_n = -eZ/A$ . (Weird, no? Think about it.) This yields the *dipole EWSR*

$$\mathcal{S}[d_z] \equiv \sum_f E_{fi} |d_{fi}^z|^2 = \frac{\hbar^2 e^2}{2m_N} \frac{NZ}{A}, \quad (16)$$



**FIGURE 2.** Left: Coulomb excitation cross section of giant resonances in  $^{40}\text{Ca}$  projectiles hitting a  $^{238}\text{U}$  target as a function of the laboratory energy per nucleon. The dashed line corresponds to the excitation of the giant electric dipole resonance, the dotted to the electric quadrupole, and the lower line to the magnetic dipole. The solid curve is the sum of these contributions. Right: Virtual photon numbers for the electric dipole multipolarity generated by 84A MeV  $^{17}\text{O}$  projectiles incident on  $^{208}\text{Pb}$ , as a function of the center-of-mass scattering angle. The solid curve is a semiclassical calculation. The dashed and dotted curves are eikonal calculations with and without relativistic corrections, respectively.

where  $m_N$  is the nucleon mass. The factor  $(NZ/A)$  is connected to the reduced mass for relative motion of neutrons against protons as required at the fixed center of mass.

The EWSR (16) is what we need to evaluate the sum of dipole cross sections for real photons over all possible final states  $|f\rangle$ . Taking the photon polarization vector along the  $z$ -axis, we obtain the total dipole photoabsorption cross section

$$\sigma_{tot}^\gamma = \sum_f \int dE_\gamma \sigma_{fi}^\gamma = 2\pi^2 \frac{e^2 \hbar}{m_N c} \frac{ZN}{A}. \quad (17)$$

This universal prediction on average agrees well with experiments in spite of crudeness of approximations made in the derivation. One should remember that it includes only dipole absorption. For the E2 isoscalar giant quadrupole resonances one can derive the approximate sum rule  $\int dE_\gamma \sigma_{GQR}^\gamma(E_\gamma)/E_\gamma^2 \simeq 0.22ZA^{2/3} \mu\text{b MeV}^{-1}$ .

**Resonances.** A simple estimate of Coulomb excitation of giant resonances based on sum rules can be made by assuming that the virtual photon numbers vary slowly compared to the photonuclear cross sections around the resonance peak. Then

$$\sigma_C \simeq \frac{n_{E1}(E_{GDR})}{E_{GDR}} \int dE_\gamma \sigma_{GDR}^\gamma(E_\gamma) + n_{E2}(E_{GQR}) E_{GQR} \int \frac{dE_\gamma}{E_\gamma^2} \sigma_{GQR}^\gamma(E_\gamma). \quad (18)$$

In figure 2 we show the Coulomb excitation cross section of giant resonances in  $^{40}\text{Ca}$  projectiles hitting a  $^{238}\text{U}$  target as a function of the laboratory energy per nucleon. The dashed line corresponds to the excitation of the giant electric dipole resonance, the dotted to the electric quadrupole, and the lower line to the magnetic dipole which was also

obtained using a sum-rule for M1 excitations [2]. The solid curve is the sum of these contributions. The cross sections increase very rapidly to large values, which are already attained at intermediate energies ( $\sim 100$  MeV/nucleon).

As with giant dipole resonances (GDR) in stable nuclei, one believes that *pygmy resonances* at energies close to the threshold are present in halo, or neutron-rich, nuclei. The *hydrodynamical model* predicts [3] for the width of the collective mode  $\Gamma = \hbar \bar{v}/R$ , where  $\bar{v}$  is the average velocity of the nucleons inside the nucleus. This relation can be derived by assuming that the collective vibration is damped by the incoherent collisions of the nucleons with the walls of the nuclear potential well during the vibration cycles (*piston model*). Using  $\bar{v} = 3v_F/4$ , where  $v_F = \sqrt{2E_F/m_N}$  is the Fermi velocity, with  $E_F = 35$  MeV and  $R = 6$  fm, one gets  $\Gamma \simeq 6$  MeV. This is the typical energy width a giant dipole resonance state in a heavy nucleus. In the case of neutron-rich light nuclei  $\bar{v}$  is not well defined. There are two average velocities: one for the nucleons in the core,  $\bar{v}_c$ , and another for the nucleons in the skin, or halo, of the nucleus,  $\bar{v}_h$ . Following Ref. [BM93], the width of momentum distributions of core fragments in knockout reactions,  $\sigma_c$ , is related to the Fermi velocity of halo nucleons by  $v_F = \sqrt{5\sigma_c^2}/m_N$ . Using this expression with  $\sigma_c \simeq 20$  MeV/c, we get  $\Gamma \simeq 1$  MeV, in accordance with experiments. Usually such modes are studied with the random phase approximation (RPA).

**Eikonal waves.** The free-particle wavefunction  $\psi \sim e^{i\mathbf{k}\cdot\mathbf{r}}$  becomes “distorted” in the presence of a potential  $V(\mathbf{r})$ . The distorted wave can be calculated numerically by performing a partial wave-expansion solving the Schrödinger equation for each partial wave, i.e., if  $\psi = \sum_{lm}(\chi_l(r)/r)Y_{lm}(\hat{\mathbf{r}})$ , then

$$\left[ \frac{d^2}{dr^2} + k_l^2(r) \right] \chi_l(r) = 0, \quad (19)$$

where

$$k_l(r) = \left\{ \frac{2\mu}{\hbar^2} \left[ E - V(r) - \frac{l(l+1)\hbar^2}{2\mu r^2} \right] \right\}^{1/2}. \quad (20)$$

with the condition that asymptotically  $\psi(\mathbf{r})$  behaves as a plane wave.

The solution of (19) involves a great numerical effort at large bombarding energies  $E$ . Fortunately, at large energies  $E$  a very useful approximation is valid when the excitation energies  $\Delta E$  are much smaller than  $E$  and the nuclei (or nucleons) move in forward directions, i.e.,  $\theta \ll 1$ . Calling  $\mathbf{r} = (z, \mathbf{b})$ , where  $z$  is the coordinate along the beam direction, we can assume that  $\psi(\mathbf{r}) = e^{ikz} \phi(z, \mathbf{b})$ , where  $\phi$  is a slowly varying function of  $z$  and  $b$ , so that  $|\nabla^2 \phi| \ll k |\nabla \phi|$ . In cylindrical coordinates the Schrödinger equation for  $\psi$  becomes

$$2ik e^{ikz} \frac{\partial \phi}{\partial z} + e^{ikz} \frac{\partial^2 \phi}{\partial z^2} + e^{ikz} \nabla_b^2 \phi - \frac{2m}{\hbar^2} V e^{ikz} \phi = 0$$

or, neglecting the 2nd and 3rd terms, we get  $\partial \phi / \partial z = -iV(\mathbf{r})/\hbar v \phi$ , whose solution is

$$\phi = \exp \left\{ -\frac{i}{\hbar v} \int_{-\infty}^z V(\mathbf{b}, z') dz' \right\}. \quad (21)$$

That is,

$$\psi(\mathbf{r}) = \exp\{ikz + i\chi(\mathbf{b}, z)\}. \quad (22)$$

This is the *eikonal function*, where

$$\chi(\mathbf{b}, z) = -\frac{1}{\hbar v} \int_{-\infty}^z V(\mathbf{b}, z') dz' \quad (23)$$

is the *eikonal phase*. Given  $V(\mathbf{r})$  one needs a single integral to determine the scattering wave. Do you have any idea how many people made their lives from the eikonal waves?. Well, don't ask, don't tell. By the way, some people call anything carrying an eikonal wavefunction by "Glauber" theory.

**Quantum scattering.** Defining  $\mathbf{r}$  as the separation between the center of mass of the two nuclei and  $\mathbf{r}'$  as the intrinsic coordinate of the target nucleus, the inelastic scattering amplitude to first-order is given by [5]

$$f(\theta) = \frac{ik}{2\pi\hbar v} \int d^3r d^3r' \left\langle \Phi_{\mathbf{k}'}^{(-)}(\mathbf{r}) \phi_f(\mathbf{r}') \mid \mathcal{H}_{int}(\mathbf{r}, \mathbf{r}') \mid \Phi_{\mathbf{k}}^{(+)}(\mathbf{r}) \phi_i(\mathbf{r}') \right\rangle, \quad (24)$$

where  $\Phi_{\mathbf{k}'}^{(-)}(\mathbf{r})$  and  $\Phi_{\mathbf{k}}^{(+)}(\mathbf{r})$  are the incoming and outgoing distorted waves, respectively, and  $\phi(\mathbf{r}')$  is the intrinsic nuclear wavefunction of the target nucleus. Looks complicated. But that is the way we calculate quantum scattering amplitudes. Sometimes one calls this the Distorted Wave Born approximation (DWBA).

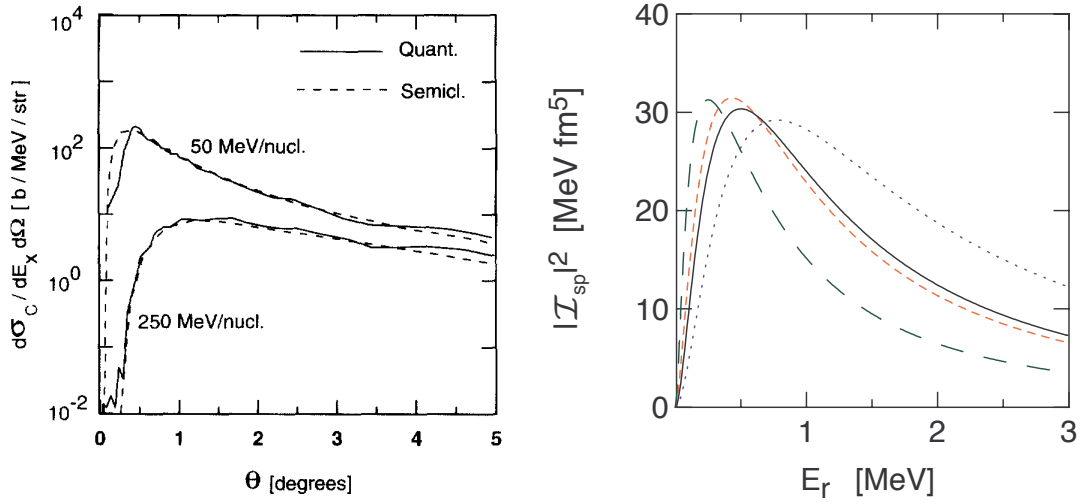
At intermediate energies,  $\Delta E/E_{lab} \ll 1$ , and forward angles,  $\theta \ll 1$ , we can use eikonal wavefunctions for the distorted waves. Corrections due to the extended nuclear charges can also be easily incorporated [5]. The results can also be cast in the form of Eq. (12). Trust me on this one.

In figure 2 we show the virtual photon numbers for the electric dipole multipolarity generated by 84A MeV  $^{17}\text{O}$  projectiles incident on  $^{208}\text{Pb}$ , as a function of the center-of-mass scattering angle. The solid curve is a semiclassical calculation. The dashed and dotted curves are eikonal calculations with and without relativistic corrections, respectively (relativity? Well, no space to explain it here. Next school.). One sees that the diffraction effects arising from the quantum treatment of the scattering change considerably the differential cross sections. The corrections of relativity are also important. However, for small excitations both semiclassical and quantum scattering yield similar results for the differential cross section, as shown in figure 3 (left).

*Single particle and collective response.* Assume a loosely-bound particle described by an Yukawa of the form  $\exp(-\eta r)/r$ , where  $\eta = \sqrt{2\mu_{bc}S/\hbar}$ ,  $\mu_{bc}$  is the reduced mass of (particle b + core c),  $S$  is the separation energy. This is a reasonable assumption for the deuteron and also for other neutron *halo* systems. We further assume that the final state is a plane-wave state (i.e., we neglect final state interactions)  $\psi_f \equiv \langle \mathbf{q} | \mathbf{r} \rangle = e^{i\mathbf{q} \cdot \mathbf{r}}$ . The response functions for electric multipole transitions, calculated from Eq. (15) is

$$\frac{dB(EL; E_\gamma)}{dE_\gamma} \sim \frac{\sqrt{S}(E_\gamma - S)^{L+1/2}}{E_\gamma^{2L+2}}. \quad (25)$$





**FIGURE 3.** Left: Comparison of quantum (solid) and semiclassical (dashed) calculations of Coulomb excitation by  $E_x = 1.5$  MeV of  $^8\text{B}$  projectiles incident on lead at 50 and 250 MeV/nucleon, respectively. Right: Square of the integrals over coordinate,  $|\mathcal{I}_{sp}|^2$ , used in the calculation of the response function (15) in the single-particle model for a light, weakly-bound nucleus. Different scattering lengths and effective ranges were assumed for the final state.

The maximum of this function occurs at  $E_0^{EL} = (L + \frac{1}{2})S / (L + \frac{3}{2}) \sim S$ .

In figure 3 (right) we show a similar calculation as described above, but accounting for final state interactions in the form of scattering lengths and effective ranges (I wish I had more space to explain that, too.). The integrals over coordinate, denoted by  $\mathcal{I}_{sp}$  show a strong dependence on the *final state interactions*. The strong dependence of the response function on the effective range expansion parameters makes it an ideal tool to study the scattering properties of light nuclei which are of interest for nuclear astrophysics.

**The Coulomb dissociation method.** As discussed above, the Coulomb breakup cross section for  $a + A \rightarrow b + c + A$  can be written as

$$\frac{d\sigma_C^{\pi L}(E_\gamma)}{d\Omega} = N^{\pi L}(E_\gamma; \theta) \sigma_{\gamma+a \rightarrow b+c}^{\pi L}(E_\gamma), \quad (26)$$

where  $E_\gamma$  is the energy transferred from the relative motion to the breakup, and  $\sigma_{\gamma+a \rightarrow b+c}^{\pi L}(E_\gamma)$  is the photo-dissociation cross section for the multipolarity  $\pi L$  and photon energy  $E_\gamma$ . Time reversal allows one to deduce the radiative capture cross section  $b + c \rightarrow a + \gamma$  from  $\sigma_{\gamma+a \rightarrow b+c}^{\pi L}(E_\gamma)$ , i.e.,

$$\sigma_{b+c \rightarrow \gamma+a} = \frac{2(2j_a + 1)}{(2j_b + 1)(2j_c + 1)} \frac{k_\gamma^2}{k^2} \sigma_{\gamma+a \rightarrow b+c}, \quad (27)$$

where  $k_\gamma = E_\gamma / \hbar c$  is the photon wavenumber, and  $k = \sqrt{2\mu(E_\gamma - B)} / \hbar$  is the wavenumber for the relative motion of  $b+c$ . Except for the extreme case very close to the threshold ( $k \rightarrow 0$ ), we have  $k_\gamma \ll k$ , so that the phase space favors the photodisintegration cross

section as compared to the radiative capture. Direct measurements of the photodisintegration near the break-up threshold do hardly provide experimental advantages and seem presently impracticable. On the other hand the copious source of virtual photons acting on a fast charged nuclear projectile when passing the Coulomb field of a (large  $Z$ ) nucleus offers a way to study cross sections close to the breakup threshold.

This method was introduced in Ref. [6] and has been tested successfully in a number of reactions of interest to astrophysics. The most celebrated case is the reaction  ${}^7\text{Be}(p, \gamma){}^8\text{B}$  (see figure 4, left). This reaction is important because it produces  ${}^8\text{B}$  in the core of our sun. These nuclei decay by emitting high energy neutrinos which are one of the best probes of the sun's interior. The measurement of such neutrinos is very useful to test our theoretical solar models.

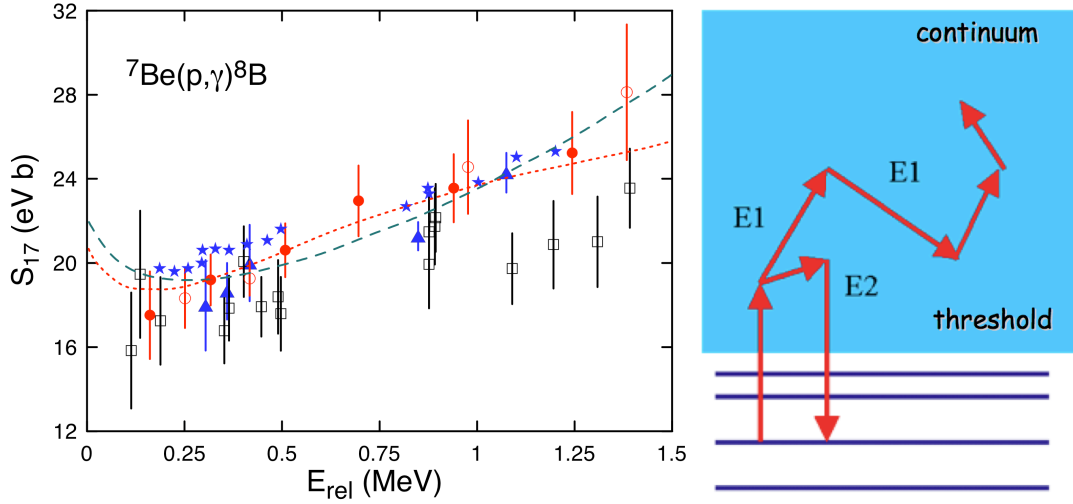
**Semiclassical CDCC.** The Coulomb dissociation method is specially useful if first-order perturbation theory is valid. If not, one can still extract the electromagnetic matrix elements involved in radiative capture reactions. But a much more careful analysis of the high-order effects needs to be done. This can be accomplished by using a time-dependent discrete states are defined as

$$\begin{aligned} |\phi_b\rangle &= e^{-iE_b t/\hbar} |b\rangle, \quad \text{for bound states} \\ \text{and } |\phi_{jJM}\rangle &= e^{-iE_j t/\hbar} \int \Gamma_j(E) |E, JM\rangle \quad \text{for continuum states,} \end{aligned} \quad (28)$$

where  $|E, JM\rangle$  are continuum wavefunctions of the projectile fragments (with or without the interaction with the target), with good energy and angular momentum quantum numbers  $E, JM$ . The functions  $\Gamma_j(E)$  are assumed to be strongly peaked around an energy  $E_j$  in the continuum. Therefore, the discrete character of the states  $|\phi_{jJM}\rangle$  (together with  $|\phi_b\rangle$ ) allows an easy implementation of the coupled-states calculations (see fig. 4, right). Calling them all together by  $|\alpha\rangle$ , the orthogonality of the discrete states (28) is guaranteed if  $\int dE \Gamma_\alpha(E) \Gamma_\beta(E) = \delta_{\alpha\beta}$ . Writing the time-dependent Schrödinger equation for  $\Psi(t) = \sum_\alpha a_\alpha(t) \phi_\alpha$ , taking the scalar product with the basis states and using orthonormality relations, we get the coupled-channels equations (3). The problem of higher-order effects has been solved in this way for several cases. It is known as Semiclassical Continuum Discretized Coupled-Channels (S-CDCC) method. You can drop the "S" if you want.

**Schrödinger equation in a lattice.** Another treatment of higher-order effects assumes solving the Schrödinger equation directly by discretizing space and time. This equation can be solved by a finite difference method assuming that the wavefunction can be expanded in several bound and unbound eigenstates  $|\alpha\rangle$ , as before. A truncation on the sum is obviously needed. To simplify, we discuss the method for one-dimensional problems. The wave function  $\Psi_\alpha$  at time  $t + \Delta t$  is obtained from the wave function at time  $t$ , according to the algorithm [7]

$$\Psi_\alpha(t + \Delta t) = \left[ \frac{1}{i\tau} - \Delta^{(2)} + \frac{\Delta t}{2\hbar\tau} V_0 \right]^{-1} \left[ \frac{1}{i\tau} + \Delta^{(2)} - \frac{\Delta t}{2\hbar\tau} V_0 + \frac{\Delta t}{\hbar\tau} \hat{S} \right] \Psi_\alpha(t). \quad (29)$$



**FIGURE 4.** Left:  $S$ -factor for the  ${}^7\text{Be}(p,\gamma){}^8\text{B}$  reaction. The solid red circles are data obtained with the Coulomb dissociation method. The curves represent different theoretical methods for the reaction. Right: The reaction of a weakly-bound nucleus can lead to several steps of excitation in the continuum.

In this equation  $\tau = \hbar\Delta t/4\mu_{bx}(\Delta x)^2$  and  $\widehat{S}\Psi_\alpha(t) = \sum_{\alpha'} \langle \alpha | V | \alpha' \rangle \Psi_{\alpha'}(t)$ , with  $V$  being the time dependent potential, responsible for the transitions.  $V_0$  is part of  $H_0$ .

The wave functions  $\Psi_\alpha(x,t)$  are discretized in a mesh in space, with a mesh-size  $\Delta x$ . The second difference operator  $\Delta^{(2)}$  is defined as  $\Delta^{(2)}\Psi_\alpha^{(j)} = \Psi_\alpha^{(j+1)}(t) + \Psi_\alpha^{(j-1)}(t) - 2\Psi_\alpha^{(j)}(t)$ , with  $\Psi_\alpha^{(j)} \equiv \Psi_\alpha(r_j, t)$ .

The wave function calculated numerically at a very large time will not be influenced by the Coulomb field. The numerical integration can be stopped there. The continuum part of the wave function is extracted by means of the relation (and normalized to unity)

$$\Psi_c(\mathbf{r}, t) = [\Psi - \Psi_{gs} \langle \Psi_{gs} | \Psi \rangle] [1 - |\langle \Psi_{gs} | \Psi \rangle|^2]^{-1/2} \quad (30)$$

where  $\Psi_{gs}$  is the initial wave function. This wave function can be projected onto an (intrinsic) continuum state to obtain the excitation probability of the state.

**Eikonal CDCC.** To get quantum dynamical equations to treat higher-order effects, one discretizes the wavefunction in terms of the longitudinal center-of-mass momentum  $k_z$ , using the ansatz

$$\Psi = \sum_{\alpha} \mathcal{S}_{\alpha}(z, \mathbf{b}) \exp(ik_{\alpha}z) \phi_{k_{\alpha}}(\xi). \quad (31)$$

In this equation,  $(z, \mathbf{b})$  is the projectile's center-of-mass coordinate, with  $\mathbf{b}$  equal to the impact parameter.  $\phi(\xi)$  is the projectile intrinsic wavefunction and  $(k, \mathbf{K})$  is the projectile's center-of mass momentum with longitudinal momentum  $k$  and transverse momentum  $\mathbf{K}$ .

Neglecting terms of the form  $\nabla^2 \mathcal{S}_\alpha(z, \mathbf{b})$  relative to  $ik\partial_z \mathcal{S}_\alpha(z, \mathbf{b})$ , the Schrödinger (or the Klein-Gordon) equation reduces to

$$i\hbar v \frac{\partial \mathcal{S}_\alpha(z, \mathbf{b})}{\partial z} = \sum_{\alpha'} \langle \alpha | V | \alpha' \rangle \mathcal{S}_{\alpha'}(z, \mathbf{b}) e^{i(k_{\alpha'} - k_\alpha)z}. \quad (32)$$

These are the *eikonal-CDCC equations* (E-CDCC). They are much simpler to solve than the complicated low-energy CDCC equations because the  $z$  and  $b$  coordinates decouple and only the evolution on the  $z$  coordinate needs to be treated non-perturbatively. Of course, I lied and there are other complications (angular momentum coupling, etc.) hidden below the rug. If quantum field theorists can do it, why can't we?

The matrix element  $\langle \alpha | V | \alpha' \rangle$  is Lorentz invariant. Boosting a volume element from the projectile to the laboratory frame means  $d^3\xi \rightarrow d^3\xi/\gamma$ . The intrinsic projectile wavefunction is a scalar and transforms according to  $\phi_\alpha(\xi_x, \xi_y, \xi_z) \rightarrow \phi_\alpha(\xi_x, \xi_y, \gamma\xi_z)$ , while  $V$ , treated as the time-like component of a four-vector, transforms as  $V(b, z; \xi_x, \xi_y, \xi_z) \rightarrow \gamma V(b, z; \xi_x, \xi_y, \gamma\xi_z)$ . Thus, redefining the integration variable  $z$  in the laboratory as  $\xi'_z = \gamma\xi_z$  leads to the afore mentioned invariance. We can therefore calculate  $\langle \alpha | V | \alpha' \rangle$  in the projectile frame. Good Lord. That makes calculations so much easier.

The longitudinal wavenumber  $\hbar k_{\alpha c} \simeq (E^2 - M^2 c^4)^{1/2}$  also defines how much energy is gone into projectile excitation, since for small energy and momentum transfers  $k'_\alpha - k_\alpha \sim (E'_\alpha - E_\alpha)/\hbar v$ . In this limit, eq. (32) reduces to the semiclassical coupled-channels equations, Eq. (3), if one uses  $z = vt$  for a projectile moving along a straight-line classical trajectory, and changing to the notation  $\mathcal{S}_\alpha(z, b) = a_\alpha(t, b)$ , where  $a_\alpha(t, b)$  is the time-dependent excitation amplitude for a collision with impact parameter  $b$ . Isn't that great!

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