Electron Screening in Stellar Environments

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Abstract

A stellar plasma, such as in the sun, consists of ions surrounded by electron clouds. The electrons "shield" the target ion, thus lowering the Coulomb potential around the ion. This lower potential increases the nuclear fusion reaction rate by a factor known as the screening enhancement factor (SEF). We review the derivation of the potential and subsequent SEF. One incorporates a parameter for the degeneracy of the electrons which allows an accurate description of screening for different stellar environments. We derive the SEF for non-degenerate, intermediate degenerate and completely degenerate stellar environments. Further, we apply a modification of the SEF accounting for the distance at which the nuclear reaction takes place. Finally, we evaluate and graph the resulting SEFs for these different stellar environments.

1 Introduction

In his pioneering book "The Internal Constitution of the Stars," Sir Arthur Eddington described the inside of a star as a "hurly-burly of atoms, electrons, and aether waves."[1] This collection of ions and electrons at a high temperature is known as a plasma. If we assume the plasma contains completely ionized atoms that have Coulomb interactions of a lower energy than their kinetic energy, we can evaluate the internal structure of a star as an ideal gas.

This simplifies the calculation of the equation of state by utilizing statistical mechanics.

The positive ions in the stellar plasma will attract the electrons in the medium to form a negatively charged cloud around each ion. This cloud can reduce the amount of energy required for two ions to react, thus increasing the fusion reaction rate. This modification to the calculation of the reaction rate is known as the screening enhancement factor (SEF).

Beginning with Salpeter's [3] determination of the screening enhancement factor over sixty years ago, many attempts at deriving the SEF have been made and have resulted in different values based on differing assumptions and approximations. We will focus on the approach by Liolios [5] which incorporates a parameter that represents the degeneracy of the electrons in the stellar plasma. The degeneracy of the electrons can range from nondegenerate, or partially degenerate as in our sun, or completely degenerate as found in stars of higher density and temperature. Inclusion of a degeneracy parameter allows determination of the SEF for a variety of stellar compositions.

2 Ionization in the stellar plasma environment

The first and major constraint in the determination of the SEF is that the stellar plasma be completely ionized. Ionization occurs when electrons leave their orbits around the nucleus resulting in a positively charged ion and free electrons. Salpeter defined a parameter I_z as the ratio of the ionization potential for a hydrogen-like atom with charge Z to the mean thermal energy. That is,

$$I_z = \left(\frac{Ze^2}{2a_0}\right)(kT_6)^{-1}, \tag{1}$$

where e is the fundamental charge of $4.8 \times 10^{-10} statcoulombs$,[1] a_0 is the Bohr radius of the atom equal to $5.292 \times 10^{-9} centimeters$, k is the Boltzmann constant equal to $1.381 \times 10^{-16} erg/K$, and T_6 the temperature in units of

[1]We use cgs units, common in nuclear astrophysics.

million degrees Kelvin. With the constants included, the ionization parameter simplifies to

$$I_z = 0.16 \frac{Z^2}{T_6} \ll 1, \tag{2}$$

where the inequality arises in most environments of interest in nuclear astrophysics. We can call the environment completely ionized if the inequality above holds. This is the case for our sun where $T_6 > 10$ and $Z = 1$ with predominantly hydrogen ignition.

3 The degeneracy parameter

In order to determine the screening enhancement factor (SEF) and its subsequent effects on nuclear reaction rates in a stellar plasma, our calculations depend on the determination of the effective Coulomb potential around the reacting ions. We can use Poisson's equation to get the potential from the charge density. The charge density is sum of the products of the number densities of the elements in the plasma with their respective charges.

Let N_i be the number density of element (i) related to the plasma density, ρ by the formula $N_i = \rho(X_i/A_i)N_0$ where X_i is the fraction by weight (mass fraction), A_i is the mass number, and N_0 being the Avogadro's number. The total number density of free particles in the plasma is the sum of number densities for ions and electrons. Because, based on the assumption of total ionization, each atom releases all of it's electrons, there are Z_i electrons in the plasma. Thus, the number density of electrons is $N_e = \rho N_0 \sum_i Z_i(X_i/A_i)$. The global charge density for the completely ionized plasma is sum of the products of the number densities with their respective charges ($Z_i e$ for the ions and e for the electrons).

$$\rho_q = \sum_{i \neq e} N_i Z_i e - N_e e. \tag{3}$$

In a neutral plasma the value of ρ_q is zero because the positive charges balance with the negative charges.

Because the electrons and ions are fermions, we need to use the Fermi-Dirac statistical formulation for the number density as a function of the momentum(p), namely[2],

$$n_i(p) = \frac{8\pi p^2/h^3}{\exp\left[\left(\frac{p^2}{2m_i} - \mu_i\right)/kT\right] + 1}, \qquad \int_0^\infty n_i(p)dp = N_i \tag{4}$$

where h is Planck's constant ($6.626 \times 10^{-34} m^2 kg/s$), m_i is the mass of the respective ion, k is Boltzmann's constant, and μ_i is the chemical potential[2]. There is a number density for each species of ion, as well as for the electrons.

The Coulomb potential varies as a function of position within the plasma, and we call it by Φ. With the introduction of a point nucleus $Z_0 e$ in the plasma, the new number densities for ions and electrons include an adjustment to the chemical potential

$$\tilde{n}_e = \frac{8\pi p^2/h^3}{\exp\left\{\left[\frac{p^2}{2m_e} - (\mu_e + e\Phi)\right]/kT\right\} + 1} \tag{5}$$

and

$$\tilde{n}_i = \frac{8\pi p^2/h^3}{\exp\left\{\left[\frac{p^2}{2m_i} - (\mu_i - eZ_i\Phi)\right]/kT\right\} + 1} \tag{6}$$

with $e\Phi$ and $-eZ_i\Phi$ being the additional energy needed to add one electron and a point nucleus, respectively.

In quantum mechanics, an energy level is said to be degenerate when it corresponds to two or more different states of the system. The degeneracy of an electron gas is related to the electron density. We incorporate a parameter for the degeneracy of the plasma (α_e for the electrons and α_i for the ions)

$$\tilde{n}_e(p) = \frac{8\pi p^2/h^3}{\exp\{\frac{p^2}{2m_e kT} + \alpha_e\} + 1}, \tag{7}$$

$$\tilde{n}_i(p) = \frac{8\pi p^2/h^3}{\exp\{\frac{p^2}{2m_i kT} + \alpha_i\} + 1}, \tag{8}$$

[2]The chemical potential is defined as the energy required to add or remove one extra particle in the plasma.

where $\alpha_e = -(\mu_e + e\Phi)/kT$ and $\alpha_i = -(\mu_i - eZ_i\Phi)/kT$. Equations (7) and (8) will allow us to determine the number densities and subsequent screening enhancement factors for stellar environments with different degeneracy values.

Integrating the number densities from zero to infinity and multiplying by their respective charges (Z_ie for the ions and $-e$ for the electrons) will give us the total charge density of the plasma.

$$\tilde{\rho}(r) = \sum_i \tilde{N}_i(Z_ie) = \int_0^\infty \left[\sum_{i\neq e} \tilde{n}_i(p)Z_ie - \tilde{n}_e(p)e\right] dp. \tag{9}$$

Using Poisson's equation with the total charge density we can calculate the potential around the reacting ion and obtain the SEF.

Recall that the number density of each ion is the integration of the number of ions with momentum (p) going from zero to infinity,

$$N_i = \int_0^\infty n_i(p)dp = \int_0^\infty \frac{8\pi p^2/h^3}{\exp\{\frac{p^2}{2m_ikT} + \alpha_i\} + 1}dp. \tag{10}$$

Evaluation of the ion density is facilitated with the introduction of the Fermi-Dirac function of order one-half, defined as

$$F_{1/2}(a) = \int_0^\infty \frac{u^{1/2}du}{\exp(a+u)+1}.$$

With $u = p^2/2mkT$, $u^{1/2} = p/\sqrt{2mkT}$ and $du = pdp/mkT$, the number densities are obtained in terms of the Fermi-Dirac function of order one-half.

$$N_i(\alpha) = \frac{8\pi}{h^3}\sqrt{2}(mkT)^{3/2}\int_0^\infty \frac{\frac{p}{\sqrt{2mkT}}\frac{p}{mkT}dp}{\exp\{\frac{p^2}{2mkT} + \alpha\} + 1}, \tag{11}$$

and

$$N_i(\alpha) = \frac{4\pi}{h^3}(2mkT)^{3/2}\int_0^\infty \frac{u^{1/2}du}{\exp\{u+\alpha\}+1}, \tag{12}$$

with the number density becoming

$$N_i(\alpha) = \frac{4\pi}{h^3}(2m_ikT)^{3/2}F_{1/2}(\alpha), \tag{13}$$

with the degeneracy parameter, α, being the variable in the function, taking on different values depending on the nature of the stellar plasma.

4 Non-degenerate, weak screened environment

Let us designate $\Lambda = \exp(-\alpha)$. For $\alpha_i > 0$ then Λ is less than unity and corresponds to a non-degenerate environment. Then the Fermi-Dirac one-half integral can be written as

$$F_{1/2} = \int_0^\infty \frac{u^{1/2}du}{(1/\Lambda)e^u + 1} = \int_0^\infty \Lambda e^{-u} u^{1/2} \frac{1}{1 + \Lambda e^{-u}} du \tag{14}$$

The term $(1 + \Lambda e^{-u})^{-1}$ can be expanded into a binomial series

$$F_{1/2}(\Lambda) = \Lambda \int_0^\infty e^{-u} u^{1/2} [1 - \Lambda e^{-u} + (\Lambda e^{-u})^2 - (\Lambda e^{-u})^3 + \ldots] du \tag{15}$$

which can be integrated term by term

$$F_{1/2}(\alpha_i) = \int_0^\infty e^{-\alpha} e^{-u} u^{1/2} - \int_0^\infty e^{-2\alpha} e^{-2u} u^{1/2} + \int_0^\infty e^{-3\alpha} e^{-3u} u^{1/2} - \ldots \tag{16}$$

to give

$$F_{1/2}(\alpha_i) = -\frac{\sqrt{\pi}}{2} \sum_{n=1}^{\infty} \frac{(-1)^n e^{-n\alpha_i}}{n^{3/2}}. \tag{17}$$

When $\alpha_i \geq 2$ all the terms in the summation beyond the first term are very small and can be neglected. Then the Fermi-Dirac function of order one-half is simply

$$F_{1/2}(\alpha_i) = -\frac{\sqrt{\pi}}{2} e^{-\alpha_i} \tag{18}$$

Since $\alpha_e = -(\mu_e + e\Phi)/kT$ for the electrons and $\alpha_i = -(\mu_i - eZ_i\Phi)/kT$ for the ions then the exponent can be separated yielding for the electrons

$$F_{1/2}(\alpha_e) = -\frac{\sqrt{\pi}}{2} \exp\left\{\frac{\mu_e}{kT}\right\} \exp\left\{\frac{e\Phi}{kT}\right\} \tag{19}$$

and for the ions,

$$F_{1/2}(\alpha_i) = -\frac{\sqrt{\pi}}{2} \exp\left\{\frac{\mu_i}{kT}\right\} \exp\left\{-\frac{eZ_i\Phi}{kT}\right\}. \tag{20}$$

We can see that the number density around the point nucleus which incorporates this approximation of the Fermi-Dirac one-half integral is simply the original number density (N_e and N_i) times an exponential factor,

$$N_e(r) \simeq N_e(a_e) \exp\left(\frac{e\Phi(r)}{kT}\right) \tag{21}$$

and

$$N_i(r) \simeq N_i(a_i) \exp\left(\frac{-eZ_i\Phi(r)}{kT}\right) \tag{22}$$

where $a_k = \mu_k/kT$ is the degeneracy parameter without correction for the charge of the point nucleus.

If the environment is weakly screened with

$$\frac{e\Phi(r)}{kT} \ll 1, \qquad \frac{Z_i e\Phi(r)}{kT} \ll 1, \tag{23}$$

then we can expand the exponential function into a power series with the higher order terms discarded,

$$N_e(r) \simeq N_e(a_e)\left(1 + \frac{e\Phi(r)}{kT}\right), \; N_i(r) \simeq N_i(a_i)\left(1 - \frac{Z_i e\Phi(r)}{kT}\right). \tag{24}$$

The charge density is the sum of the number densities times their respective charges. With the charge density in Poisson's equation, $\nabla^2\Phi = -4\pi\rho_c$, we can get the potential, $\Phi(r)$, around the reacting ion,

$$\nabla^2\Phi(r) = -4\pi e\left[\sum_{i\neq e} N_i(a_i)\left(1 - \frac{Z_i e\Phi(r)}{kT}\right) Z_i - N_e(a)\left(1 + \frac{e\Phi(r)}{kT}\right)\right]. \tag{25}$$

Plasma neutrality implies that the charge of the ions cancel the charge of the electrons resulting in the elimination of two terms in the Poisson's equation. One gets

$$\nabla^2\Phi(r) = \frac{4\pi e^2}{kT}\left(\sum_{i\neq e} N_i(a_i)Z_i^2 + N_e(a)\right)\Phi(r). \tag{26}$$

On the right-hand side of the equation, we can group all of the components that are multiplied by the potential variable $\Phi(r)$ into a single unit known as the Debye radius R_D:

$$R_D^{-2} = \frac{4\pi e^2}{kT}\left(\sum_{i\neq e} N_i(a_i)Z_i^2 + N_e(a)\right). \tag{27}$$

The Poisson equation can be expressed in a single radial coordinate (from the location of the ion). That is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = \frac{1}{R_D^2}\Phi(r). \tag{28}$$

Then moving the r^2 from the left side to the right side we get a simple differential equation

$$\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = \left(\frac{r}{R_D}\right)^2 \Phi(r). \tag{29}$$

The solution to this differential equation is

$$\Phi(r) = A\exp(-r/R_D), \tag{30}$$

where A is a normalization constant. If we add the important boundary condition that, as r goes to zero, $\Phi(r)$ goes to $Z_i e/r$, we can solve for the constant A to get the final equation for the Debye-Huckel Coulomb potential around the reacting ion to be

$$\Phi(r) = \frac{Z_i e}{r}\exp(-r/R_D), \tag{31}$$

with

$$\Phi(0) = \frac{Z_i e}{r} - \frac{Z_i e}{R_D}. \tag{32}$$

Salpeter [3] originally used the Debye-Hueckel Coulomb potential just derived to determine the screening enhancement factor for a non-degenerate, weakly screened stellar plasma environment. The nuclear fusion reaction rate for unscreened nuclei is proportional to the following integral,

$$\int_0^\infty dE[E^{1/2}e^{-E/kT}]P(E)\sigma(E), \tag{33}$$

where the second term is the barrier penetration factor ($P(E)$).

There is a critical distance (r_c) at which the energy E_{max} in the collision between two nuclei of charge Z_1 and Z_2 is defined by

$$E_{max} = \frac{Z_1 Z_2 e^2}{r_c}. \tag{34}$$

P(E) at this critical distance, r_c, is dependent on the expression

$$\left[E - U(r_{12}) - \frac{Z_1 Z_2 e^2}{r_{12}}\right], \tag{35}$$

where the first term is the relative kinetic energy between the two nuclei, the second term is the adjustment of energy from the screening electrons, and the third term is the potential energy of the reacting nuclei. The function $U(r_{12})$ is small for $r_{12} \gg R_D$ and approaches U_0 as r_{12} approaching zero. But if the inequality

$$\frac{r_c}{R_D} \sim \frac{U_0}{E_{max}} \ll 1 \tag{36}$$

holds; then we can replace $U(r_{12})$ with U_0. The nuclear fusion reaction rate integral now becomes

$$\int_0^\infty dE[(E+U_0)^{1/2} e^{-E/kT} e^{-U_0/kT}] P(E)\sigma(E). \tag{37}$$

Because U_0 is small, $(E+U_0)^{1/2}$ can be approximated as $E^{1/2}$. Now we have the original integral (33) for unscreened nuclei including a factor

$$e^{-U_0/kT}. \tag{38}$$

Since $U_0 = Z_2 e\Phi(0)$ where $\Phi(0)$ is defined by equation (32) we have the solution for the screening enhancement factor is

$$f_s = \exp \frac{Z_1 Z_2 e^2}{R_D kT} \tag{39}$$

with R_D defined by equation (27).

5 Completely degenerate stellar environment

In section 3 the number density of the electrons and ions incorporated the Fermi-Dirac function of order one-half with the degeneracy parameter of α. For a non-degenerate environment we used a value of $\alpha \geq 2$. Now we can set α to a large negative number for a completely degenerate environment. Recall the incorporation of the parameter $\Lambda = \exp(-\alpha)$ in the Fermi-Dirac function of one-half,

$$F_{1/2} = \int_0^\infty \frac{u^{1/2} du}{(1/\Lambda)e^u + 1}. \tag{40}$$

If we take the variable $u^{1/2}$ to be the derivative of a function $\phi(u)$ that is a sufficiently regular function which vanishes for $u = 0$ then we can use the

Sommerfield-Lemma (Appendix A) to expand the Fermi-Dirac function of one-half [2]:

$$\int_0^\infty \frac{du}{(1/\Lambda)e^u + 1}\frac{d\phi(u)}{du} = \phi(u_0) + 2\left[c_2\left(\frac{d^2\phi}{du^2}\right)_{u_0} + c_4\left(\frac{d^4\phi}{du^4}\right)_{u_0} + \ldots\right] \quad (41)$$

where $u_0 = \ln \Lambda$ and $c_2,\ c_4,\ \ldots$ are numerical constants defined by

$$c_\nu = 1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \frac{1}{4^\nu} + \ldots$$

with the summations equal to

$$c_2 = \frac{\pi^2}{12} \qquad c_4 = \frac{7\pi^4}{720} \qquad c_6 = \frac{31\pi^6}{30,240}.$$

Applying the Sommerfield-Lemma to the Fermi-Dirac function of one-half yields the following expansion (Appendix B)

$$F_{1/2}(\alpha) = \frac{2}{3}(-\alpha)^{3/2}\left(1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4}\ldots\right). \quad (42)$$

Replacing α with $\alpha + y$ where $y = -e\Phi/kT > 0$ we have

$$F_{1/2}(\alpha) = \frac{2}{3}[-(a+y)]^{3/2}\left(1 + \frac{\pi^2}{8}(a+y)^{-2} + \frac{7\pi^4}{640}(a+y)^{-4} + \ldots\right). \quad (43)$$

Each summation in parentheses raised to a power can be expressed in a binomial expansion $(a+b)^n = a^n + na^{n-1}b$ with the discarding of higher order terms. This results in a revised equation for the Fermi Dirac function of order one-half to be (Appendix C)

$$F_{1/2}(\alpha) \simeq F_{1/2}(a)\left[1 + \theta(a)\frac{e\Phi}{kT}\right], \quad (44)$$

with

$$\theta(a) = \frac{3}{2}(-a)^{-1} - \frac{2}{3}\frac{(-a)^{1/2}}{F_{1/2}(a)}\left(\frac{\pi^2}{4a^2} + \frac{7\pi^4}{160a^4}\right). \quad (45)$$

When the degeneracy parameter $a < -3$ and $0 < \theta(a) < 1$ then we can simplify $\theta(a)$ to (Appendix D)

$$\theta(a) = -\frac{5}{2a}\frac{384a^4 - 16\pi^2a^2 - 7\pi^4}{640a^4 + 80\pi^2a^2 + 7\pi^4}. \quad (46)$$

We can further simplify $\theta(a)$ by taking only terms larger than the term with a^{-2} giving (Appendix E)

$$\theta(a) = -\frac{1}{2a}\left[\frac{24a^2 - \pi^2}{8a^2 + \pi^2}\right] \tag{47}$$

which gives results for large negative degeneracy parameters. Setting the limit of the degeneracy parameter to negative infinity gives the simple equation

$$\lim_{a \to -\infty} \theta(\alpha) = \frac{3}{2}(-\alpha)^{-1}. \tag{48}$$

Our resulting equations for theta can be compared to Mitler's derivation [4]

$$\theta(\alpha) = \left[1 + \frac{4}{9}\left[\frac{3}{2}F_{1/2}(\alpha)\right]^{4/3}\right]^{-1/2}. \tag{49}$$

Figure 1 illustrates the value of theta resulting from the different derived equations. The thin solid line represents the value of theta from eqn.(42). The dashed line shows theta from eqn.(43). The dotted line presents theta for eqn.(44). The dash-dot line gives theta from eqn.(45). Finally, the solid line gives theta from Mitler's equation (46).

This theta function is incorporated into the electron number density to yield

$$N_e(r) = N_e(a)\left[1 + \theta(a)\frac{e\Phi}{kT}\right], \tag{50}$$

which is used in the Poisson's equation to get the potential around the reacting ion resulting in a new Debye radius of

$$R_D^{-2} = \frac{4\pi e^2}{kT}\left(\sum_{i \neq e} N_i(a_i) Z_i^2 + \theta(a) N_e(a)\right). \tag{51}$$

This Debye radius, modified by the inclusion of the parameter $\theta(a)$ for completely degenerate electrons, is used in the screening enhancement factor such that

$$f_s = \exp\left(\frac{Z_1 Z_0 e^2}{kT R_D(a)}\right) \tag{52}$$

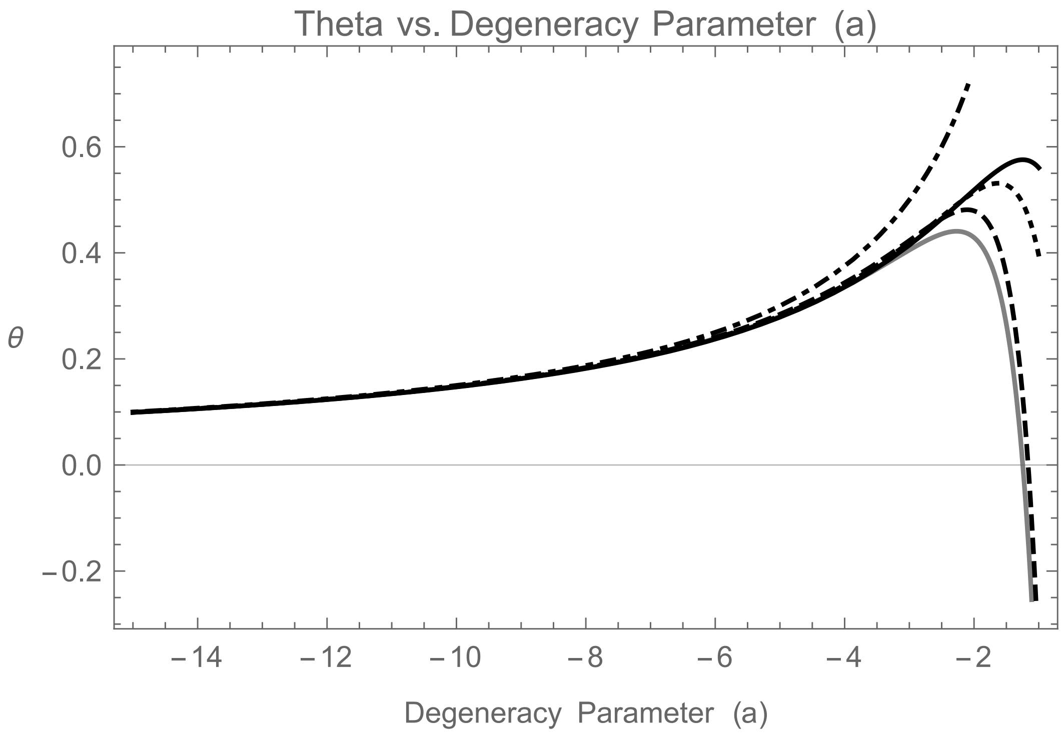

Figure 1: Evaluation of theta using different approximations.

6 Degeneracy effects in different stellar environments

From equation (13) and equating the stellar electron density, ρ and mean molecular weight per electron, μ_e we have

$$\frac{\rho N_0}{\mu_e} = N_e(\alpha) = \frac{4\pi}{h^3}(2m_e kT)^{3/2} F_{1/2}(\alpha) \tag{53}$$

where $N_0 = 6.022 \times 10^{23}\ mole^{-1}$, $h = 6.626 \times 10^{-27}\ erg - sec$, $m_e = 9.109 \times 10^{-28}\ grams$, and $k = 1.381 \times 10^{-16}\ erg/deg(K)$. Incorporating the constants and taking the logarithm we get the value of the degeneracy parameter, α is related to the stellar environment by

$$\log_{10}\left(\frac{\rho}{\mu_e} T^{-3/2}\right) = \log_{10} \frac{F_{1/2}(\alpha)}{9.04887 \times 10^{-9}} = \log_{10} F_{1/2}(a) - 8.044. \tag{54}$$

Then we can write the inequality for $(a \geq a^*)$ of

$$\frac{\rho}{\mu_e} \leq 10^{9+Q(a^*)} T_6^{3/2}, \tag{55}$$

where $Q(a^*) = \log_{10} F_{1/2}(a^*) - 8.044$ and T_6 is million degrees of temperature T, ($T_6 = T \times 10^6$).

For the boundary between non-degenerate and partially degenerate plasma environments we set $a^* = 2$ we have

$$\log_{10} F_{1/2}(2) - 8.044 = \log_{10} 0.11994 - 8.044 \cong -9, \tag{56}$$

and we get the inequality of

$$\frac{\rho}{\mu_e} \leq 10^{(9-9)} T_6^{3/2} = T_6^{3/2}. \tag{57}$$

This defines the criteria for the non-degenerate environment ($\alpha \geq 2$) with the use of an uncorrected factor while evaluation of the partially degenerate stellar plasma ($-3 \leq \alpha < 2$) must incorporate the correction of the $\theta(\alpha)$ factor given by Salpeter [3] as

$$\theta(\alpha) = F_{1/2}^{-1}(\alpha) \frac{dF_{1/2}(\alpha)}{d\alpha}. \tag{58}$$

For a strongly degenerate stellar plasma environment in which $\alpha \leq -5$ we use equation (39) with $\alpha = -5$,

$$F_{1/2}(-5) = \frac{2}{3}(5)^{3/2} \left(1 + \frac{\pi^2}{8 * (-5)^2} + \frac{7\pi^4}{640 * (-5)^4}\right) \tag{59}$$

resulting in $F_{1/2}(-5) = 7.8227$ and $\log_{10} F_{1/2}(-5) = 0.8934$. Thus

$$10^{(9+.8934-8.044)} \simeq 71 \tag{60}$$

and

$$\frac{\rho}{\mu_e} \geq 71 T_6^{3/2}. \tag{61}$$

This equation represents the criteria for a stellar environment that is intermediate degenerate. For a completely degenerate environment with $\alpha \leq -10$ we again use equation (39) and set $\alpha = -10$ resulting in $F_{1/2}(-10) = 21.3422$ and $\log_{10} F_{1/2}(-10) = 1.3292$ with

$$10^{(9+1.3292-8.044)} \simeq 193. \tag{62}$$

Thus

$$\frac{\rho}{\mu_e} \geq 193T_6^{3/2}. \tag{63}$$

This equation illustrates the criteria for a stellar environment that is completely degenerate. The dividing line between non-degenerate and completely degenerate stellar environment is determined by equating the non-degenerate electron pressure to the completely degenerate electron pressure,

$$N_0kT\left(\frac{\rho}{\mu_e}\right) = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{2/3}(N_0)^{5/3}\left(\frac{\rho}{\mu_e}\right)^{5/3}. \tag{64}$$

Incorporating the constants gives

$$8.3145 \times 10^7 T = 1.004 \times 10^{13}\left(\frac{\rho}{\mu_e}\right)^{2/3}, \tag{65}$$

resulting in

$$\frac{\rho}{\mu_e} = 24T_6^{3/2}. \tag{66}$$

This equation represents the point at which the the stellar environment transitions from non-degenerate to completely degenerate.

Salpeter's SEF is valid in a plasma consisting of Hydrogen and Helium with zero additional metals. Recalling inequality (23) we have

$$\frac{Z_iZ_0e^2}{\langle r\rangle_i}\exp\left(-\frac{\langle r\rangle_i}{R_D}\right) \ll kT. \tag{67}$$

In this plasma the inter-ionic distances between $H - H$ and $H - He$ are of the same order of magnitudes as the inter-electronic distances. That is $\langle r\rangle_{ee} \sim n_e^{-1/3}$ with $n_e = \rho N_0/\mu_e$. If we apply the constants of $e = 4.8032 \times 10^{-10}$ *statcoulombs*, $N_0 = 6.0221 \times 10^{23}$ $mole^{-1}$, $k = 1.3807 \times 10^{-16}$ $erg/deg(K)$ and approximate the exponent to be an exponent of a small number and equal to one, then for the Hydrogen ion

$$\frac{\rho}{\mu_e} \ll 350T_6^3, \tag{68}$$

and for the Helium ion with $Z_i = 2$,

$$\frac{\rho}{\mu_e} \ll \frac{350}{8}T_6^3. \tag{69}$$

These equations represent the range at which the first derivation of the SEF are valid.

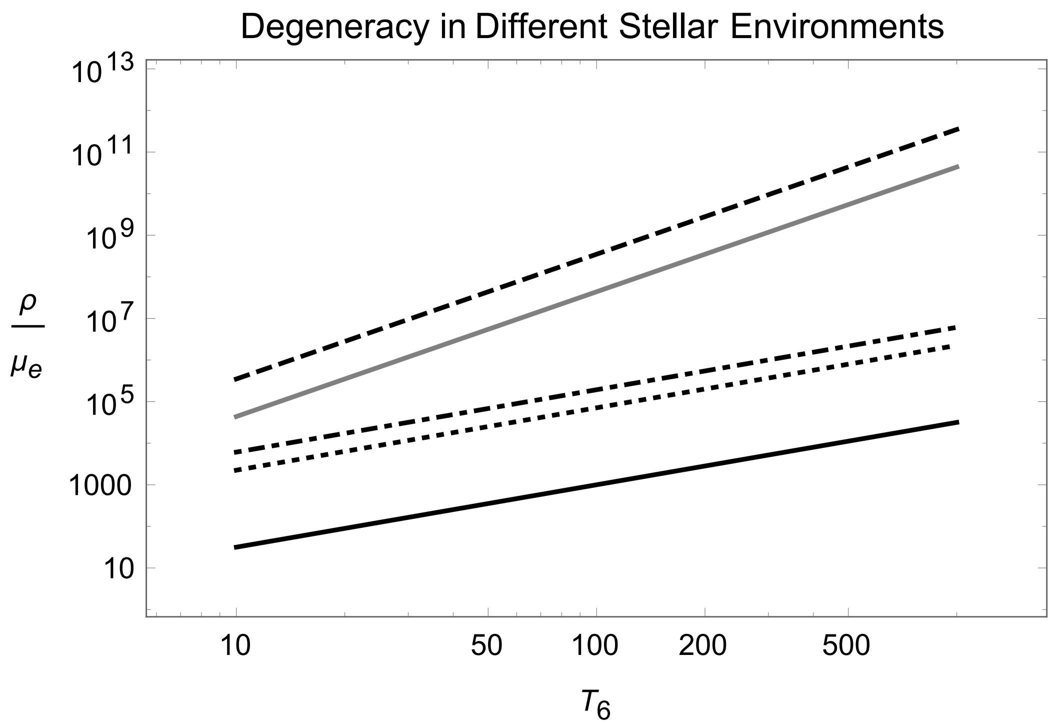

Figure 2: Lines demarcating degeneracy in different stellar environments.

Figure 2 shows the lines of demarcation of the degeneracies within the different stellar environments. Below the solid black line we evaluate the screening enhancement factor using the non-degenerate parameter ($\alpha \geq 2$). Above the dotted line indicates the stellar environment in which an intermediate degeneracy is used ($\alpha \leq -5$). Complete degeneracy ($\alpha \leq -10$) is evaluated in the stellar environments indicated above the dash-dotted line. In a zero-metalicity environment the H-H SEF can be determined in a stellar environment indicated by the dashed line. Finally, the H-He SEF can be evaluated in the stellar environment indicated by the upper solid line.

7 A modified screening enhancement factor

The previous screening enhancement factors for non-degenerate and completely degenerate environments rely on a Debye radius and a Debye radius with a theta incorporated, respectively. However, the nuclear fusion reactions take place at an inter-ionic distance that is much shorter than the Debye length. We need to modify the screening enhancement factor to account for reactions at this shorter distance.

7.1 Nonlinear sreening effect

For the Debye-Huckel Coulomb potential, $\Phi_{DH}(r)$ to be valid in the tunneling region of the WKB integral used in the penetration factor the same inequalities used to derive the Debye-Huckel potential should be valid within that region,

$$\frac{Z_i e \Phi_{DH}(r)}{kT} \ll 1 \quad \text{that is} \quad \frac{Z_i Z_0 e^2}{kTr} \exp\left(-\frac{r}{R_D}\right) \ll 1. \tag{70}$$

Equating $x = r/R_D$ we can rearrange the inequality to form

$$\frac{e^{-x}}{x} \ll (\ln f_{max})^{-1} \tag{71}$$

where

$$f_{max} = \exp\left(\frac{Z_{max} Z_0 e^2}{R_D kT}\right) \tag{72}$$

is Salpeter's screening factor for the heaviest ion interacting with the central nucleus of $Z_0 e$. The condition that $f_{max} > 1$ indicates that there is an x_{min} such that $x \gg x_{min}$. That means there is a critical value x_0 below which the previously derived Debye-Huckel screening enhancement factors are not valid. We can convert the inequality to an equation from which we can solve for that minimum value x_0. When we incorporate a tuning factor, β, in the equation, the inequality is still valid and we have

$$\frac{e^{x_0}}{x_0} = \beta(\ln f_s^{max})^{-1}. \tag{73}$$

We can use the value of β to tune the equation for specific plasma environments and setting $\beta = 0.1$ provides a minimal one percent error in the linearization of the number densities used to obtain the Debye-Huckel potential.

7.2 Evaluation of critical distance and the central density

In our previously derived screening enhancement factors we assumed the charge density to be constant and equivalent to the average charge density of the electrons. We will now take the approach that the charge density around the nucleus is an exponentially decreasing function of distance of the form

$$\rho_{in}(r) = \rho(0) \exp(-r/R_0) \tag{74}$$

where R_0 is a new value replacing R_D.

At the critical point r_0 the density inside of r_0 is equal to the density outside of r_0 and

$$\rho_{in}(r_0) = \rho_{out}(r_0) = \left(-\frac{Z_0 e}{4\pi R_D^2}\right) \frac{\exp(-r_0/R_D)}{r_0}. \tag{75}$$

Let $x_0 = r_0/R_D$ and equation (75) becomes

$$\rho_{in}(x_0) = \rho_{out}(x_0) = \left(-\frac{Z_0 e}{4\pi R_D^3}\right) \frac{\exp(-x_0)}{x_0}. \tag{76}$$

Since

$$\left(-\frac{Z_0 e}{4\pi R_D^3}\right) = \rho_{out}(R_D) \exp(1), \tag{77}$$

we can substitute equation (77) into equation (76) to get the density at r_0 in terms of the density at the Debye radius

$$\rho_{in}(x_0) = \rho_{out}(x_0) = \frac{e^{1-x_0}}{x_0} \rho_{out}(R_D). \tag{78}$$

Rearranging equation (74) we get that the central charge density is defined as

$$\rho(0) = \rho_{in}(x_0) \exp\left(x_0'\right), \tag{79}$$

Substituting the definition of the density inside r_0 being equation (78) into the definition of the central charge density (79) results in an equation that includes the values of x_0 and x'_0,

$$\rho(0) = \rho_{in}(x_0) \exp(x'_0) = \frac{e^{x'_0 - x_0}}{x_0} \left(-\frac{Z_0 e}{4\pi R_D^3} \right), \tag{80}$$

where $x_0 = r_0/R_D$ and $x'_0 = r_0/R_0$.

The charge normalization condition states that the total charge of the system is the sum of the charge inside r_0 and the charge outside r_0,

$$\int_0^{r_0} \rho_{in}(r) 4\pi r^2 dr \; + \; \int_{r_0}^{\infty} \rho_{out}(r) 4\pi r^2 dr \; = \; -Z_0 e. \tag{81}$$

Inserting the definitions of the densities we have

$$4\pi\rho(0) R_0^3 \int_0^{x'_0} x^2 e^{-x} dx - Z_0 e \int_{x_0}^{\infty} x e^{-x} dx = -Z_0 e. \tag{82}$$

These integrals are trivial and can be solved analytically

$$I_1(x'_0) = \int_0^{x'_0} x^2 e^{-x} dx = -e^{-x'_0} \left(x_0'^2 + 2x'_0 + 2 \right) + 2 \tag{83}$$

and

$$I_2(x_0) = \int_{x_0}^{\infty} x e^{-x} dx = e^{-x_0} (x_0 + 1). \tag{84}$$

Then equation (82) becomes

$$4\pi\rho(0) R_0^3 I_1(x'_0) - Z_0 e I_2(x_0) = -Z_0 e. \tag{85}$$

Combining eqns. (74),(77),(78) and (79) we get the equation which can be solved for x'_0 given x_0,

$$\frac{e^{x'_0}}{x_0'^3} I_1(x'_0) = \frac{e^{x_0}}{x_0^2} [1 - I_2(x_0)]. \tag{86}$$

7.3 Derivation of the screened Coulomb potential

Once the inner charge density has been derived, we can obtain the screened Coulomb potential from Poisson's equation

$$\nabla^2 \Phi = -4\pi\rho(0) \exp(-r/R_0), \tag{87}$$

with the solution in the form of

$$\Phi(r) = \frac{Z_0 e}{r} F(r). \tag{88}$$

Applying spherical symmetry and boundary conditions we obtain the function

$$F(r) = c_2 + c_1 r - \frac{4\pi\rho(0) R_0^3}{Z_0 e}\left(2 + \frac{r}{R_0}\right) e^{-r/R_0}, \tag{89}$$

with

$$c_2 = 1 - 2\frac{x_0^2}{x_0'^3} e^{x_0' - x_0}, \tag{90}$$

and

$$c_1 = -R_D^{-1}\left[\frac{1}{x_0} - \frac{e^{-x_0}}{x_0} - 2\frac{x_0}{x_0'^3} e^{x_0' - x_0} + \frac{x_0}{x_0'^3}\left(2 + x_0'\right) e^{-x_0}\right]. \tag{91}$$

7.4 Derivation of the screening enhancement factor

Mitler's derivation of the screening enhancement factor was greatly facilitated by the concept that the charge density was constant in the region of the nuclear interaction which allowed an analytical solution. However, our charge density is more complicated and the derivation of the screening enhancement factor will be developed under certain assumptions.

a) $Z_0 e$ and $Z_1 e$ are symmetric and the impinging nucleus $Z_1 e$ is considered unscreened, which means it carries no electron cloud. As it collides with ion $Z_0 e$ it encounters the screening cloud which creates the potential $\Phi(r)$.

b) We consider the charge $Z_0 e$ is significantly larger than $Z_1 e$. Consequently, the electron cloud around $Z_0 e$ is much larger than the cloud around $Z_1 e$ and we can disregard the screening effect from $Z_1 e$. In such a case, the classical turning point lies so deeply within the screening configuration that it makes no difference whether the cloud is attributed to $Z_0 e$ or $Z_1 e$.

Because the classical turning point is much smaller than R_0 we can truncate the Coulomb potential to

$$\Phi(r) = \frac{Z_0 e}{r} - \frac{Z_0 e}{R_D} G(x_0, x_0') + O(r^2), \tag{92}$$

where the quantity $G(x_0, x_0')$ is given by

$$G(x_0, x_0') = \frac{1}{x_0} - \frac{e^{-x_0}}{x_0} - 2\frac{x_0}{x_0'^3} e^{x_0' - x_0} + \frac{x_0}{x_0'^3}(2 + x_0')e^{-x_0} + \frac{x_0}{x_0'^2} e^{x_0' - x_0}. \qquad (93)$$

Consequently, the screening energy is shifted by the function $G(x_0, x_0')$ with the new energy value related to the screening energy of the previously derives factors as

$$U_0' = U_0 G\left(x_0, x_0'\right) \qquad (94)$$

and using the definition of Salpeter's screening enhancement factor, the new factor is

$$f = e^{\frac{-U_e G(x_0, x_0')}{kT}} = \exp\left(-\frac{U_e}{kT}\right)^{G(x_0, x_0')} = f_s^{G(x_0, x_0')}. \qquad (95)$$

8 Evaluation of the derived SEFs

We have derived the screening enhancement factor (SEF) for stellar environments that are non-degenerate and completely degenerate. Additionally, we have formulated a new SEF taking in the consideration that the nuclear reaction takes place at a distance much closer than the Debye radius. We will now evaluate the SEF for each of these derivations.

8.1 SEF for a non/partially degenerate stellar environments

Recall that the SEF is determined by the equation

$$f_s = \exp\left(\frac{Z_1 Z_0 e^2}{kT R_D}\right), \qquad (96)$$

where the Debye radius is

$$R_D^{-2} = \frac{4\pi e^2}{kT}\left(\sum_{i \neq e} N_i(a_i) Z_i^2 + N_e(a)\right), \qquad (97)$$

for a non-degenerate environment.

Having $N_i = \rho(X_i/A_i)N_0$ and $N_e = \rho N_0 \sum_i (X_i/A_i)$ we can expand the equation for the SEF with the Debye radius to

$$f_s = \exp\left(\frac{Z_1 Z_0 e^2}{kT}\sqrt{\frac{4\pi e^2 \rho N_0}{kT}\left(\sum_{i\neq e}(X_i/A_i)Z_i^2 + \sum_i (X_i/A_i)\right)}\right). \tag{98}$$

For simplification we assume a stellar environment consisting of only Hydrogen with zero metalicity. Then $Z_1, Z_0, X_i, A_i = 1$ and incorporating the constants the equation for the SEF becomes

$$f_s = \exp\left(0.188\rho^{1/2}T_6^{-3/2}\right). \tag{99}$$

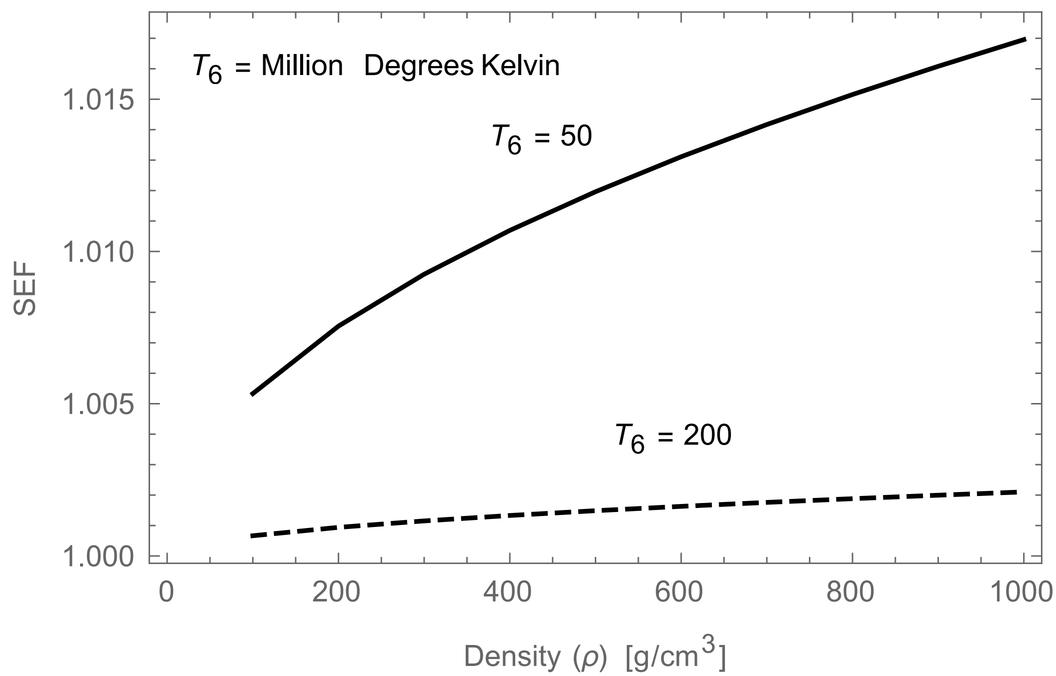

Figure 3: SEF for non-degenerate and partially degenerate stellar environments.

In figure 3, we display the SEF for a non-degenerate and partially degenerate environments in which the density is low and the temperature is high. The solid line shows a partially degenerate environment with $T_6 = 50\ K$ and the dashed line illustrates a non-degenerate environment with $T_6 = 200\ K$. Recall that $T_6 = T \times 10^6 K$. We can see that the SEF approaches one as the stellar environment approaches non-degeneracy.

8.2 SEF for intermediate/completely degenerate stellar environments

A stellar environment in which the density is high and the temperature is low results in an intermediate or completely degenerate system. We use the same equation for the SEF in a completely degenerate environment as that for the non-degenerate stellar environment. However, the Debye radius includes the function of theta,

$$R_D^{-2} = \frac{4\pi e^2}{kT}\left(\sum_{i \neq e} N_i(a_i) Z_i^2 + \theta(a) N_e(a)\right), \tag{100}$$

with theta evaluated by

$$\theta(a) = -\frac{5}{2a}\frac{384a^4 - 16\pi^2 a^2 - 7\pi^4}{640a^4 + 80\pi^2 a^2 + 7\pi^4}. \tag{101}$$

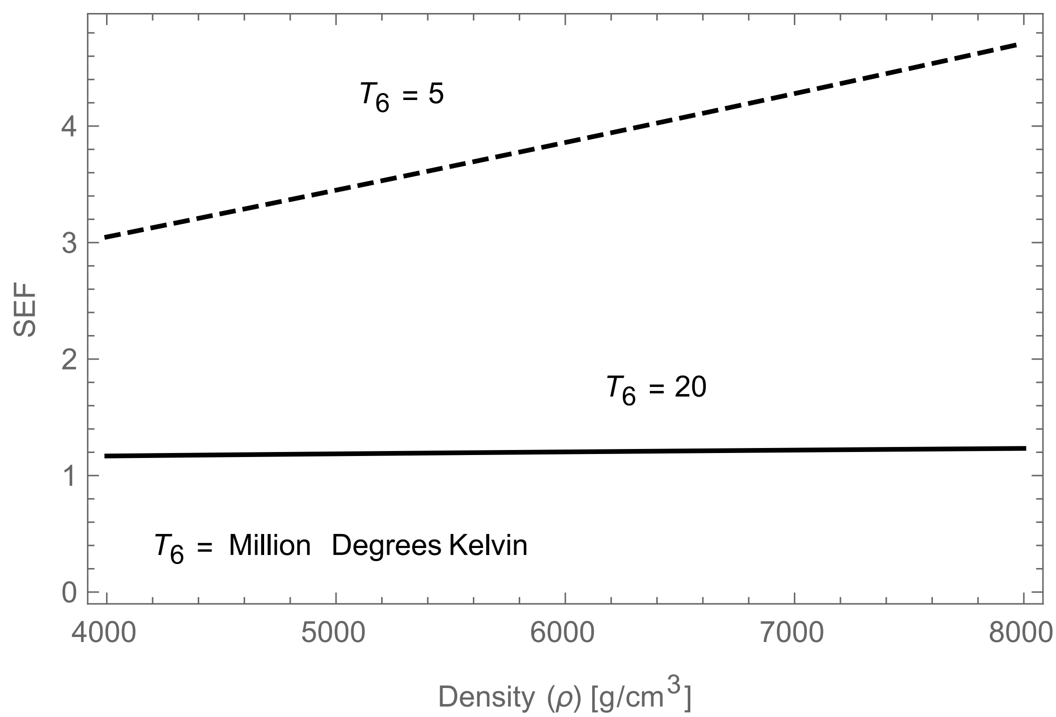

Figure 4: SEF for intermediate and completely degenerate stellar environments.

Just as in the evaluation of the SEF for a non-degenerate stellar environment we assume all hydrogen with zero metalicity and combine the constants to give the SEF as

$$f_s(\rho, T_6) = \left[\exp\left(0.188\rho^{1/2}T_6^{-3/2}\right)\right]^{\sqrt{1+\theta(\rho,T_6)}}. \tag{102}$$

Figure 4 shows the evaluation of the SEF with the solid line being an intermediate degenerate environment and the dashed line showing the completely degenerate system. The screening enhancement factor is significant in a stellar environment that is completely degenerate.

8.3 Revised SEF for different stellar environments

Derivation of the previous SEFs did not take into consideration the fact that the nuclear reaction takes place at a distance much shorter than the Debye radius. A modified SEF was derived with this correction. The original screening enhancement factor was derived as

$$f_s = e^{U_{DH}}. \tag{103}$$

The revised SEF involves a shift in the screening energy

$$U_e = U_{DH} G\left(x_0, x_0'\right) = \frac{Z_0 Z_1 e^2}{R_D} G\left(x_0, x_0'\right) \tag{104}$$

with $G(x_0, x_0')$ given by equation (88). The function $G(x_0, x_0')$ incorporates the values of x_0 and x_0' which are obtained from eqns. (70) and (81). This makes the new SEF defined as

$$f = e^{U_{DH} G(x_0, x_0')} = f_s^{G\left(x_0, x_0'\right)}. \tag{105}$$

The power of $G(x_0, x_0')$ ranges from zero to one depending on the degeneracy with non-degenerate being closer to one.

The final results are shown in figures 5-8 with the solid line being the original SEF and the dashed line representing the revised SEF. The high temperature and low density produces a non-degenerate and partially degenerate stellar environment and the SEF is hardly modified by the factor $G(x_0, x_0')$. As the temperature decreases and the density increases the SEF is reduced by the influence of the revised reaction distance.

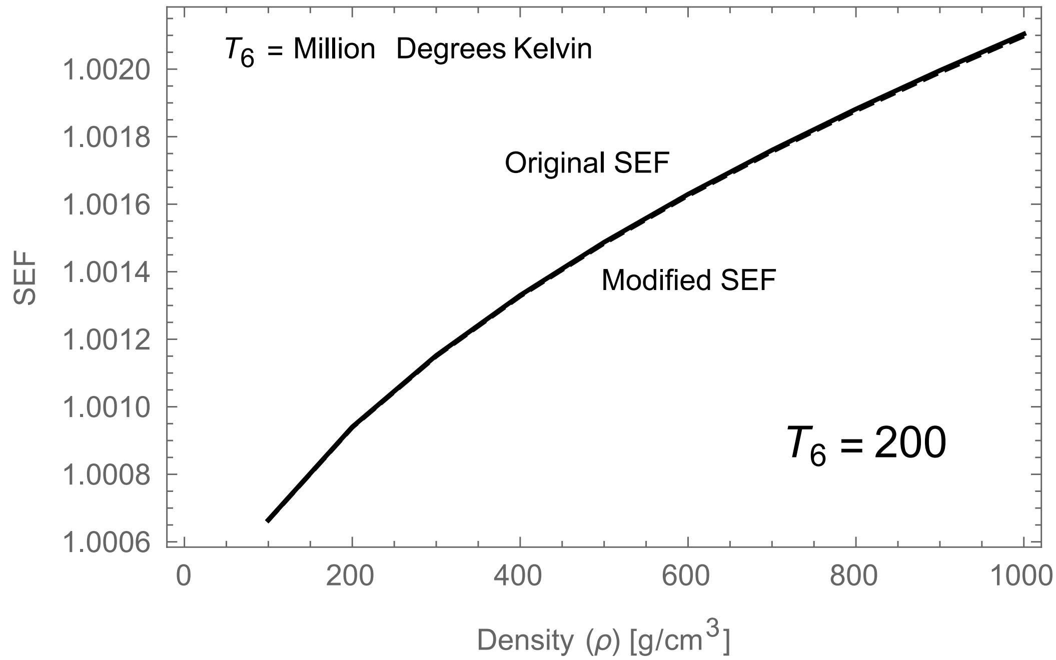

Figure 5: Revised SEF for non-degenerate stellar environment.

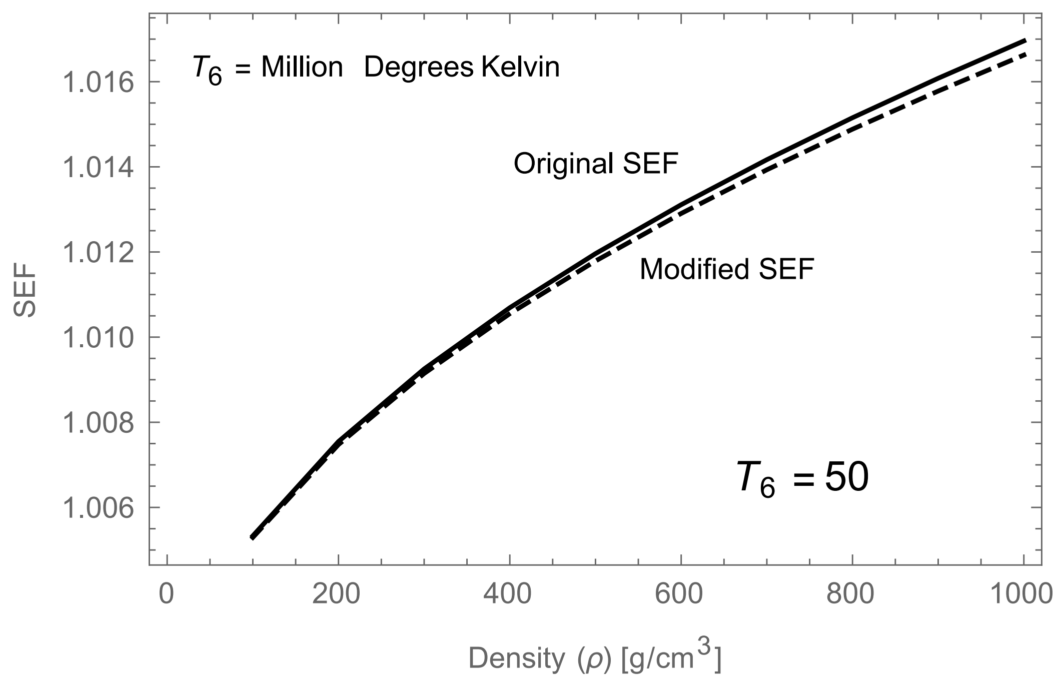

Figure 6: Revised SEF for partially degenerate stellar environment.

9 Conclusion

In stellar environments such as the sun, the plasma consists of ions and electrons. Because of their charge, each ion is surrounded by a cloud of electrons.

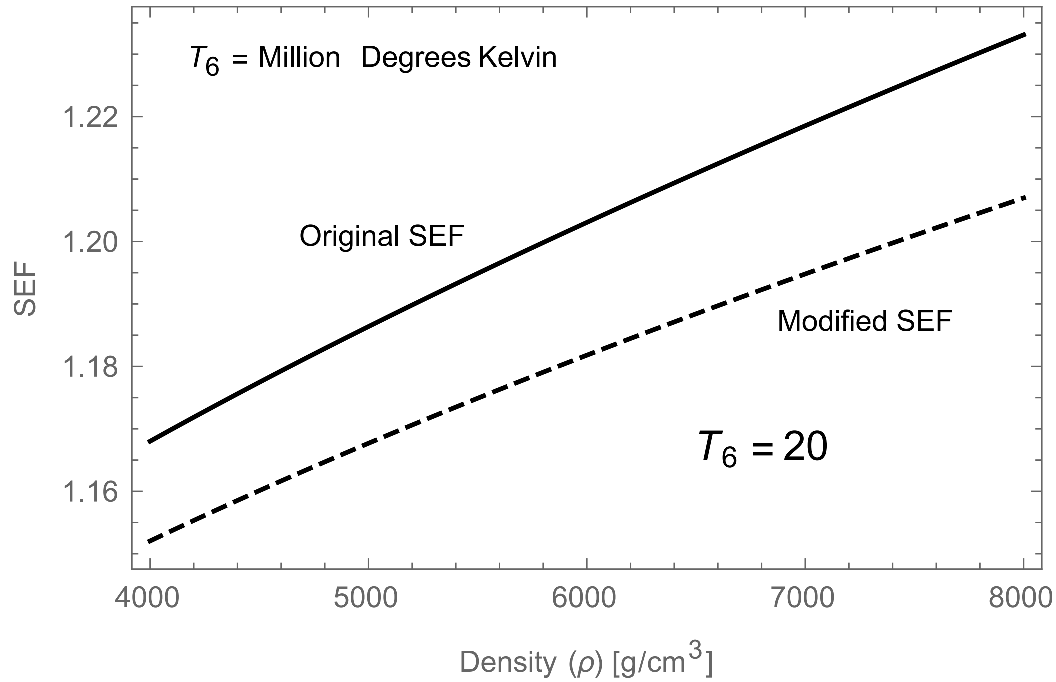

Figure 7: Revised SEF for intermediate degenerate stellar environment.

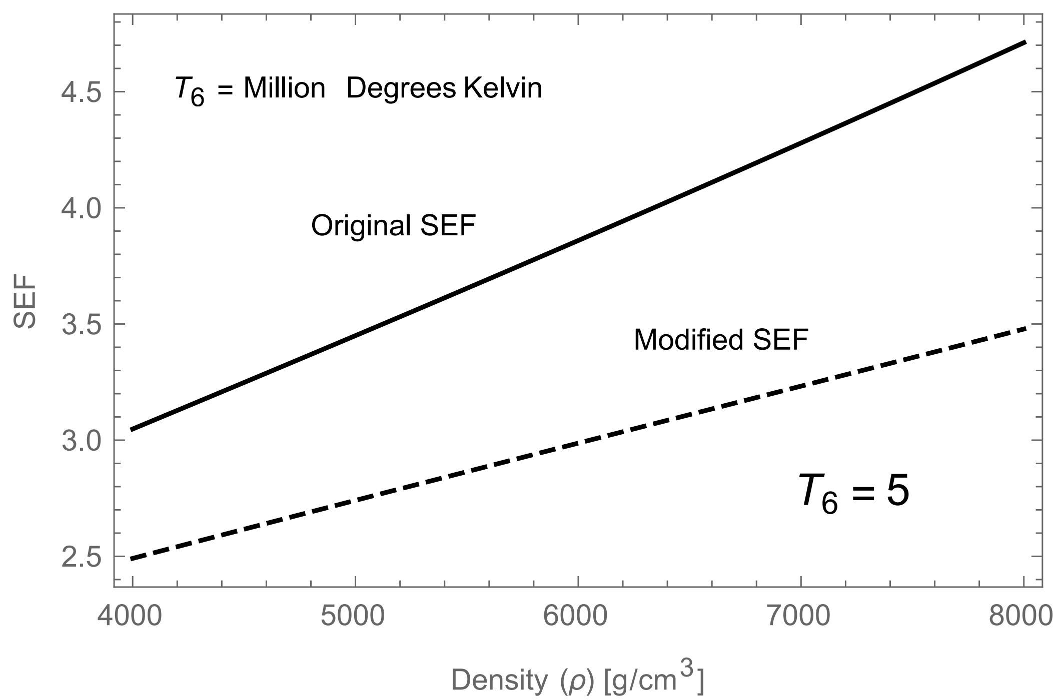

Figure 8: Revised SEF for completely degenerate stellar environment.

This negatively charged cloud "screens" the fusion reaction between two ions and alters the reaction rate by a factor known as the screening enhancement factor (SEF). We have derived the screening enhancement factor for several different stellar environments.

The derivation of the SEF involves the determination of the charge density around the reacting ion. From the charge density we obtain the Coulomb potential which includes a factor known as the Debye radius. The Debye radius is the distance at which the electron cloud has a significant effect on the Coulomb potential and is a critical distance for the SEF.

We have derived the SEF, which includes a factor for the Debye radius, for a non-degenerate stellar environment. Evaluation of the SEF shows that in this environment the factor is close to one and the reaction rate is not altered much.

For intermediate and completely degenerate stellar environments the SEF includes a Debye radius that incorporates a function θ. This factor accounts for the degeneracy of the electron cloud. Our evaluation of the SEF in this environment indicates that the reaction rate is significantly increased by the SEF.

Finally, we have revised the SEF with consideration for the fact that the reaction takes place at a distance much smaller than the Debye radius. The modified SEF results from the original SEF raised to a factor that includes this critical distance. Our evaluation of the revised SEF indicates that as the degeneracy goes from non-degenerate to completely degenerate, the SEF is reduced from its originally derived SEF.

The study presented in this thesis clearly points to the necessity of going beyond the old Salpeter's [3] theory of electron screening. It can have an important impact on the calculation of fusion reactions in stars. More work in this direction is worthwhile.

A Proof of Sommerfeld Lemma [6]

$$\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} = \int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} + \frac{d\phi(u)}{du} - \frac{d\phi(u)}{du} \tag{106}$$

$$\begin{aligned}\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} &= \int_0^{u_0} \frac{d\phi(u)}{du}du + \int_0^{u_0}\left(\frac{1}{\frac{1}{\Lambda}e^u+1}-1\right)\frac{d\phi(u)}{du}du \\ &+ \int_{u_0}^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du}\end{aligned} \tag{107}$$

$$\begin{aligned}\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} &= \phi(u_0) - \int_0^{u_0} \frac{du}{1+\Lambda e^{-u}}\frac{d\phi(u)}{du} \\ &+ \int_{u_0}^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du}.\end{aligned} \tag{108}$$

Now let $u = u_0(1-t)$ in the first integral on the right hand side of equation (104) and $u = u_0(1+t)$ in the second integral on the right hand side of equation (104). and having $u_0 = \log\Lambda$,

$$\begin{aligned}\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} &= \phi(u_0) - u_0\int_0^1 \frac{\phi'[u_0(1-t)]}{1+e^{u_0 t}}dt \\ &+ u_0\int_0^\infty \frac{\phi'[u_0(1+t)]}{1+e^{u_0 t}}dt.\end{aligned} \tag{109}$$

Now we extend the range of the first integral on the right hand side of equation (105) to infinity which adds an error of the order of e^{-u_0}, which is beyond the range of accuracy for the asymptotic formula we are establishing. Hence, we have

$$\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} \simeq \phi(u_0) + u_0\int_0^\infty \frac{\phi'[u_0(1+t)]-\phi'[u_0(1-t)]}{1+e^{u_0 t}}dt \tag{110}$$

$$= \phi(u_0) + 2\sum_{\nu=2,4,6,\ldots} \frac{u_0^\nu \phi^{(\nu)}(u_0)}{(\nu-1)!}\int_0^\infty \frac{t^{\nu-1}}{1+e^{u_0 t}}dt. \tag{111}$$

However the integral in equation (107) can be expanded:

$$\int_0^\infty \frac{t^{\nu-1}}{1+e^{u_0 t}}dt = \int_0^\infty t^{\nu-1}(e^{-u_0 t} - e^{-2u_0 t} + e^{-3u_0 t} - \ldots)dt \tag{112}$$

$$= \frac{(\nu-1)!}{u_0^\nu}\left(1 - \frac{1}{2^\nu} + \frac{1}{3^\nu} - \ldots\right). \tag{113}$$

Inserting the integral equivalent of equation (109) into equation (107) yields

$$\int_0^\infty \frac{du}{\frac{1}{\Lambda}e^u+1}\frac{d\phi(u)}{du} \simeq \phi(u_0)+2[c_2\phi''(u_0)+c_4\phi^{(iv)}(u_0)+\ldots], \tag{114}$$

$$c_2 = 1-\frac{1}{2^2}+\frac{1}{3^2}-\ldots=\frac{\pi^2}{12}, \tag{115}$$

$$c_4 = 1-\frac{1}{2^4}+\frac{1}{3^4}-\ldots=\frac{7\pi^4}{720}, \tag{116}$$

$$c_6 = 1-\frac{1}{2^6}-\frac{1}{3^6}-\ldots=\frac{31\pi^6}{30,240}. \tag{117}$$

B Sommerfeld Lemma Applied to Fermi Dirac

Let $d\phi(u)/du = u^{1/2}$, $\Lambda = \exp(-\alpha)$ and $u_0 = \log\Lambda$ then

$$\phi(u_0) = \frac{2}{3}(-\alpha)^{3/2}, \tag{118}$$

$$\left(\frac{d^2\phi}{du^2}\right)_{u_0} = \frac{1}{2(-\alpha)^{1/2}} = \frac{2}{3}(-\alpha)^{3/2}\frac{3}{4\alpha^2}, \tag{119}$$

$$\left(\frac{d^4\phi}{du^4}\right)_{u_0} = \frac{3}{8(-\alpha)^{5/2}} = \frac{2}{3}(-\alpha)^{3/2}\frac{9}{16\alpha^4}. \tag{120}$$

$$\int_0^\infty \frac{du}{(1/\Lambda)e^u+1}\frac{d\phi(u)}{du} = \phi(u_0)+2\left[c_2\left(\frac{d^2\phi}{du^2}\right)_{u_0}+c_4\left(\frac{d^4\phi}{du^4}\right)_{u_0}+\ldots\right] \tag{121}$$

$$= \frac{2}{3}(-\alpha)^{3/2}+2\left(\frac{\pi^2}{12}\frac{3}{4\alpha^2}\frac{2}{3}(-\alpha)^{3/2}\right)+2\left(\frac{7\pi^4}{720}\frac{9}{16\alpha^4}\frac{2}{3}(-\alpha)^{3/2}\right)+\ldots \tag{122}$$

$$= \frac{2}{3}(-\alpha)^{3/2}\left[1+\frac{\pi^2}{8\alpha^2}+\frac{7\pi^4}{640\alpha^4}+\ldots\right] \tag{123}$$

C Derivation of Fermi Dirac with Theta

With $y = -(e\Phi/kT)$ and replacing α with $\alpha + y$ we begin

$$F_{1/2}(\tilde{\alpha}) \simeq \frac{2}{3}[-(\alpha+y)]^{3/2}\left[1+\frac{\pi^2}{8}(\alpha+y)^{-2}+\frac{7\pi^4}{640}(\alpha+y)^{-4}\right]. \tag{124}$$

Using binomial expansion of the terms with exponents we get

$$\simeq \frac{2}{3}\left[(-a)^{3/2}+\frac{3}{2}(-a)^{1/2}(-y)\right]\left[1+\frac{\pi^2}{8}(a^{-2}-2a^{-3}y)+\frac{7\pi^4}{640}(a^{-4}-4a^{-5}y)\right]. \tag{125}$$

Multiplying out the terms in the second half of the equation gives

$$F_{1/2}(\tilde{\alpha}) \simeq \left[\frac{2}{3}(-a)^{3/2} - (-a)^{1/2}y\right]\left[1 + \frac{\pi^2}{8a^2} - \frac{\pi^2}{4a^3}y + \frac{7\pi^4}{640a^4} - \frac{7\pi^4}{160a^5}y\right]. \tag{126}$$

Distributing the factors results in four components,

$$F_{1/2}(\tilde{\alpha}) \simeq \tfrac{2}{3}(-\alpha)^{3/2}\left[1 + \tfrac{\pi^2}{8\alpha^2} + \tfrac{7\pi^4}{640\alpha^4}\right] \tag{127}$$

$$+\tfrac{2}{3}(-\alpha)^{3/2}\left[-\tfrac{\pi^2}{4\alpha^3}y - \tfrac{7\pi^4}{160\alpha^5}y\right] \tag{128}$$

$$-(-\alpha)^{1/2}y\left[1 + \tfrac{\pi^2}{8\alpha^2} + \tfrac{7\pi^4}{640\alpha^4}\right] \tag{129}$$

$$-(-\alpha)^{1/2}y\left[-\tfrac{\pi^2}{4\alpha^3}y - \tfrac{7\pi^4}{160\alpha^5}y\right]. \tag{130}$$

Now puttting in $y = -(e\Phi/kT)$.

$$F_{1/2}(\tilde{\alpha}) \simeq \tfrac{2}{3}(-\alpha)^{3/2}\left[1 + \tfrac{\pi^2}{8\alpha^2} + \tfrac{7\pi^4}{640\alpha^4}\right] \tag{131}$$

$$-\tfrac{2}{3}(-\alpha)^{1/2}\left[\tfrac{\pi^2}{4\alpha^2} + \tfrac{7\pi^4}{160\alpha^4}\right]\tfrac{e\Phi}{kT} \tag{132}$$

$$+(-\alpha)^{1/2}\left[1 + \tfrac{\pi^2}{8\alpha^2} + \tfrac{7\pi^4}{640\alpha^4}\right]\tfrac{e\Phi}{kT} \tag{133}$$

$$-(-\alpha)^{1/2}y^2\left[-\tfrac{\pi^2}{4\alpha^3} - \tfrac{7\pi^4}{160\alpha^5}\right]. \tag{134}$$

Term number (126) is simply

$$F_{1/2}(\alpha) = \frac{2}{3}(-\alpha)^{3/2}\left[1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4}\right]. \tag{135}$$

Because it includes y^2, which approaches zero, term (129) is approximately zero.

Terms number (127) and (128) are incorporated in to the equation of theta

$$F_{1/2}(\alpha)\theta(\alpha) = (-\alpha^{1/2})\left[1 + \frac{\pi^2}{8\alpha^2} + \frac{7\pi^4}{640\alpha^4}\right]\frac{e\Phi}{kT} \tag{136}$$

$$- \frac{2}{3}(-\alpha)^{1/2}\left[\frac{\pi^2}{4\alpha^2} + \frac{7\pi^4}{160\alpha^4}\right]\frac{e\Phi}{kT} \tag{137}$$

$$= F_{1/2}(\alpha)\frac{3}{2}(-\alpha)^{-1}\frac{e\Phi}{kT} \tag{138}$$

$$- F_{1/2}(\alpha)\frac{2}{3}\frac{(-\alpha)^{1/2}}{F_{1/2}(\alpha)}\left[\frac{\pi^2}{4\alpha^2} + \frac{7\pi^4}{160\alpha^4}\right]\frac{e\Phi}{kT}, \tag{139}$$

Giving the final form of the Fermi Dirac integral of order one-half with the function theta as

$$F_{1/2}(\alpha) \simeq F_{1/2}(a)\left[1+\theta(a)\frac{e\Phi}{kT}\right], \tag{140}$$

and theta equal to

$$\theta(a) = \frac{3}{2}(-a)^{-1} - \frac{2}{3}\frac{(-a)^{1/2}}{F_{1/2}(a)}\left(\frac{\pi^2}{4a^2}+\frac{7\pi^4}{160a^4}\right). \tag{141}$$

D Simplification of Theta I

$$\theta(a) = \frac{3}{2}(-a)^{-1} - \frac{2}{3}\frac{(-a)^{1/2}}{F_{1/2}(a)}\left(\frac{\pi^2}{4a^2}+\frac{7\pi^4}{160a^4}\right) \tag{142}$$

$$\theta(a) = \frac{3}{2}(-a)^{-1} - \frac{(2/3)(-a)^{1/2}}{(2/3)(-a)^{3/2}}\left(\frac{\pi^2/4a^2+7\pi^4/160a^4}{1+\pi^2/8a^2+7\pi^4/640a^4}\right) \tag{143}$$

$$\theta(a) = -\frac{1}{a}\left[\frac{3}{2}-\left(\frac{\pi^2/4a^2+7\pi^4/160a^4}{1+\pi^2/8a^2+7\pi^4/640a^4}\right)\right] \tag{144}$$

$$\theta(a) = -\frac{1}{a}\left[\frac{3+3\pi^2/8a^2+21\pi^4/640a^4-2\pi^2/4a^2-14\pi^4/160a^4}{2+2\pi^2/8a^2+14\pi^4/640a^4}\right] \tag{145}$$

$$\theta(a) = -\frac{1}{a}\left[\frac{1920a^4/640a^4+240\pi^2a^2/640a^4+21\pi^4/640a^4-320\pi^2a^2/640a^4-56\pi^4/640a^4}{1280a^4/640a^4+160\pi^2a^2/640a^4+14\pi^4/640a^4}\right] \tag{146}$$

$$\theta(a) = -\frac{1}{a}\left[\frac{1920a^4-80\pi^2a^2-35\pi^4}{1280a^4+160\pi^2a^2+14\pi^4}\right] \tag{147}$$

$$\theta(a) = -\frac{5}{2a}\left[\frac{384a^4-16\pi^2a^2-7\pi^4}{640a^4+80\pi^2a^2+7\pi^4}\right] \tag{148}$$

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$$\theta(a) = -\frac{5}{2a}\left[\frac{384a^4-16\pi^2a^2-7\pi^4}{640a^4+80\pi^2a^2+7\pi^4}\right] \tag{149}$$

Eliminate terms lower than a^{-1}.

$$\theta(a) = -\frac{5}{2a}\left[\frac{384a^4-16\pi^2a^2}{640a^4+80\pi^2a^2}\right] \tag{150}$$

$$\theta(a) = -\frac{5}{2a}\left[\frac{(24\times 16)a^4-16\pi^2a^2}{(8\times 80)a^4+80\pi^2a^2}\right] \tag{151}$$

$$\theta(a) = -\frac{(5 \times 16)a^2}{(2 \times 80)a^3}\left[\frac{24a^2 - \pi^2}{8a^2 + \pi^2}\right] \tag{152}$$

$$\theta(a) = -\frac{1}{2a}\left[\frac{24a^2 - \pi^2}{8a^2 + \pi^2}\right]. \tag{153}$$

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