Electromagnetic interaction of ultrarelativistic heavy ions

C. A. Bertulani
Department of Physics, Brookhaven National Laboratory, Upton, New York 11973-5000
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The validity of a δ-function approximation for the electromagnetic interaction of relativistic heavy ions is investigated. The production of e^+e^- pairs, with electron capture, is used as a test of the approximation.

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The production of e^+e^- in peripheral collisions of relativistic heavy ions has attracted a great amount of theoretical interest due to its nonperturbative character. The calculations are hard to perform, and it is common to find substantial differences between the cross sections calculated within several approaches [1]. A good simplification of the problem was been found by Baltz and co-workers [2]. They showed that if one makes a gauge transformation in the wave function of the form $\psi = \exp\{-i\chi(\mathbf{r},t)\}\psi'$, where

$$\chi(\mathbf{r},t) = \frac{Z\alpha}{v} ln[\gamma(z-vt) + \sqrt{b^2+\gamma^2(z-vt)^2}], \tag{1}$$

the interaction induced by the electromagnetic field of an ultrarelativistic particle is gauge transformed to (in our units $\hbar = c = m_e = 1$)

$$V(\boldsymbol{\rho},z,t) = \phi(\boldsymbol{\rho},z,t)(1-v\hat{\alpha}_z) - \phi(\boldsymbol{\rho}=\mathbf{0},z,t)(1-\hat{\alpha}_z/v), \tag{2}$$

where $\phi(\boldsymbol{\rho},z,t)$ is the Lienard-Wiechert potential at a point $\mathbf{r}=(\boldsymbol{\rho},z)$, generated by a relativistic particle with velocity $\mathbf{v} = v\hat{\mathbf{z}}$ and impact parameter $\mathbf{b}$:

$$\phi(\boldsymbol{\rho},z,t) = \gamma Z\alpha[(\mathbf{b}-\boldsymbol{\rho})^2 + \gamma^2(z-vt)^2]^{-1/2}. \tag{3}$$

In these expressions $\gamma = (1-v^2)^{-1/2}$ is the Lorentz contraction factor, and $\hat{\alpha}_z$ is the third of the Dirac matrices.

The second part of Eq. (2) acts as a regularization term of the modified potential. It removes the divergence at $b=0$. This potential is very useful, since the Lorentz contraction yields a δ-function in the longitudinal variables when $\gamma \gg 1$, and b is not too large. Evidently, this is a great simplification since δ-function interactions always lead to a considerable decrease of integration steps in perturbative as well as nonperturbative calculations. In Ref. [2] a formal derivation of the δ-function interaction was obtained by expanding the gauge transformed potential [Eq. (2)] into multipoles. Further manipulation of the multipole expansion and comparison with numerical calculations have shown that Eq. (2) can be expressed as

$$V(\boldsymbol{\rho},z,t) = \delta(z-t) Z\alpha(1-\hat{\alpha}_z) \ln\frac{(\mathbf{b}-\boldsymbol{\rho})^2}{b^2}. \tag{4}$$

We will show that this expression can be obtained in a simpler way. The derivation is useful to study the validity of the δ-function approximation. In particular we will test the approximation in a solvable problem, namely, the production of e^+e^- pairs, in which the electron is captured in an orbit around one of the nuclei (bound-free pairs).

Using the Bethe integral [3], potential (3) can be written in the form

$$\phi(\boldsymbol{\rho},z,t) = Z\alpha\frac{1}{2\pi^2}\int d^3q \frac{e^{-i\mathbf{q}\cdot\mathbf{u}}e^{i\mathbf{q}\cdot\mathbf{r}}}{q^2 - v^2q_z^2}, \tag{5}$$

where $\mathbf{u} = \mathbf{b} + \mathbf{v}t$ and $q = (\mathbf{q}_t, q_z)$. For relativistic particles we can replace $(1-v\hat{\alpha})$ and $(1-\hat{\alpha}/v)$ by $(1-\hat{\alpha})$ in interaction (2). As shown in Ref. [3] this amounts to neglecting a very small $[\sim O(1/\gamma^2)]$ piece of the longitudinal part of the interaction. However, it is important to keep the other v factors in their respective places, as they give rise to important combinations of γ factors in the matrix elements. Moreover, the integral in Eq. (5) diverges logarithmically as $v \to 1$.

The exact interaction is then given by

$$V(\boldsymbol{\rho},z,t) = Z\alpha(1-\hat{\alpha}_z)\frac{1}{2\pi^2}\int d^3q\, e^{-i\mathbf{q}\cdot\mathbf{u}}\frac{[e^{i\mathbf{q}\cdot\mathbf{r}} - e^{iq_z z}]}{q_t^2 + q_z^2/\gamma^2}, \tag{6}$$

where the denominator of the integrand in Eq. (5) has been rewritten in terms of γ. The interaction given by Eq. (4) is a limit of this integral when we set $q_z^2/\gamma^2 = 0$. It is clear from the above equation that neglecting this factor yields the δ-function in Eq. (4). However, to emphasize the restrictions on $\mathbf{q}_t$ and q_z let us define

$$\begin{aligned}\Phi(\boldsymbol{\rho},z,t) &\equiv \frac{V(\boldsymbol{\rho},z,t)}{Z\alpha(1-\hat{\alpha}_z)} \\ &= \frac{1}{\pi}\int d^2q_t \frac{1}{q_t^2}\exp(-i\mathbf{q}_t\cdot\mathbf{b}) \\ &\quad \times [\exp(i\mathbf{q}_t\cdot\boldsymbol{\rho}) - 1]\Phi_z(\mathbf{q}_t, z, t),\end{aligned} \tag{7}$$

where

$$\Phi_z(\mathbf{q}_t,z,t) = \frac{q_t^2}{2\pi}\int dq_z \frac{e^{iq_z(z-vt)}}{q_t^2 + q_z^2/\gamma^2} = \frac{\gamma q_t}{2} e^{-\gamma q_t|z-vt|}. \tag{8}$$

Now, using $\lim_{\Lambda\to\infty}(\Lambda/2)e^{-\Lambda|x|} = \delta(x)$, we see that for $\gamma \to \infty$, Φ_z does not depend on q_t, and assumes the form of a δ-function: $\Phi_z(z,t) = \delta(z-vt)$.

In this limit, we can write Eq. (7) as

$$\Phi(\boldsymbol{\rho},z,t)=\delta(z-t)\Phi_\rho(\boldsymbol{\rho},t), \tag{9}$$

with

$$\Phi_\rho(\boldsymbol{\rho},t)=\frac{1}{\pi}\int d^2q_t\frac{1}{q_t^2}\exp(-i\mathbf{q}_t\cdot\mathbf{b})[\exp(i\mathbf{q}_t\cdot\boldsymbol{\rho})-1]$$

$$=2\int\frac{dq_t}{q_t}\{J_0(q_t|\boldsymbol{\rho}-b|)-J_0(q_tb)\}, \tag{10}$$

where J_0 is the cylindrical Bessel function. The integral over each Bessel function diverges, but their difference does not. To show this we regularize the integrals by using

$$\int dx\frac{xJ_0(ax)}{x^2+k^2}=K_0(ak), \tag{11}$$

where K_0 is the modified cylindrical Bessel function. Taking the limit $k\to0$, and using $K(ak)\simeq\ln(ak)$, for small values of ak, we obtain

$$\Phi(\boldsymbol{\rho},t)=\ln\frac{(\mathbf{b}-\boldsymbol{\rho})^2}{b^2}. \tag{12}$$

This is the solution of the Coulomb potential of a unit charge in two dimensions. An easy way to see this is to use Gausś law for the electric field in two dimensions. One obtains $E\simeq1/b$, where b is the distance to the charge. Since $E=-\partial\Phi/\partial b$, the logarithmic form of Φ is evident.

The above derivation illustrates the validity of the approximation in terms of the transverse momentum transfer q_t. It should fail for very soft processes, i.e., those for which $q_t\to0$. Also, it requires that q_z be small compared to γ. As shown in Ref. [3], q_z values in the range of one unit up to γ units of the electron mass contribute appreciably to the integrals involved in the production of free, and of bound-free, e^+e^- pairs. It is thus important to check the validity of approximation (4) in a concrete case. We will do this for the production of bound-free pairs. The full calculation uses the interaction given by Eq. (6). For comparison, a similar calculation with the term $\xi=q_z/\gamma$ replaced by zero in the denominator of Eq. (6) is equivalent to the use of the interaction (4).

In Ref. [4] it was shown that the Coulomb distortion of the positron wave function is an important effect in calculations of bound-free pair production. The right magnitude of the differential cross sections depends on this effect. This is not relevant in our case, since we are only interested in the relative change of the cross sections and probabilities by using the exact and the δ-function interaction, respectively. Therefore, for simplicity, we will use plane waves for the positron wave function and first-order perturbation theory. The energy transfer from the field to the created pair is given by $\omega=\varepsilon+1$, where ε is the positron energy, neglecting the atomic binding energy of the captured electron. Using an interaction in the form of Eq. (6), the amplitude for bound-free pair production is given by

$$a_{fi}=i\int_{-\infty}^{\infty}dt\,e^{i\omega t}\langle\psi_{e^+}|V|\psi_{e^-}\rangle=\frac{iZ\alpha}{\pi}\int d^2q_t\frac{e^{-i\mathbf{q}_t\cdot\mathbf{b}}}{q_t^2+\xi^2}F(\mathbf{q}_t), \tag{13}$$

where the integral over time yields

$$F(\mathbf{q}_t)=\int d^3r\exp\left(i\frac{\omega z}{v}\right)[\exp(i\mathbf{q}_t\cdot\boldsymbol{\rho})-1]$$

$$\times\langle\psi_{e^+}|(1-\hat{\alpha}_z)|\psi_{e^-}\rangle, \tag{14}$$

and ξ is now given by $\xi=\omega/\gamma v$.

Using plane waves for the positron wave function and a hydrogenic K-orbital function for the electron, the above matrix element is given by

$$F(\mathbf{q}_t)=f(\mathbf{q}_t)\bar{\mathrm{v}}(1-\hat{\alpha}_z)u, \tag{15}$$

where

$$f(\mathbf{q}_t)=8\sqrt{\pi}(Z\alpha)^{5/2}\left\{\frac{1}{[1/a_H^2+|\mathbf{Q}-\mathbf{p}|^2]^2}\right.$$

$$\left.-\frac{1}{[1/a_H^2+|\mathbf{Q}_0-\mathbf{p}|^2]^2}\right\}. \tag{16}$$

In these equations $a_H=1/\alpha=5.29\times10^4$ fm is the Bohr radius, $v(u)$ is the positron (electron) spinor, $p=\sqrt{\varepsilon^2-1}$ is the positron momentum, $\mathbf{Q}=(\mathbf{q}_t,\omega/v)$, and $\mathbf{Q}_0=(0,\omega/v)$. Integrating the square modulus of Eq. (13) over $\mathbf{b}$ yields a δ function $\delta(\mathbf{q}_t-\mathbf{q}_t')$. Furthermore, performing the spin averages, for the differential cross section in terms of the positron energy we obtain

$$\frac{d\sigma}{d\varepsilon d\Omega}=\frac{(Z\alpha)^2}{2\pi^3}(\varepsilon-1)p\int d^2q_t\frac{|f(\mathbf{q}_t)|^2}{(q_t^2+\xi^2)^2}. \tag{17}$$

The integral over the positron scattering angle can be done analytically, as well as the remaining integral over $\mathbf{q}_t$.

To test the δ-function interaction, we define the function

$$\Delta(\varepsilon)=[(d\sigma/d\varepsilon)_{\xi=0}-d\sigma/d\varepsilon]/(d\sigma/d\varepsilon)_{\xi=0}, \tag{18}$$

which does not depend on Z.

In Fig. 1 we plot the function $\Delta(\varepsilon)$ for SPS, RHIC, and LHC heavy ion energies. The positron energies are given in MeV units. For SPS the above formulation applies directly, assuming that the electron is captured in the target and neglecting the atomic screening effects. For RHIC and LHC a transformation of Eq. (17) back to the laboratory system was performed. We note that the δ-function interaction works very well for positron energies of the order of MeV for SPS and RHIC, and up to 100 MeV for LHC. The approximation worsens abruptly at a certain positron energy. The value of ε where this occurs is a function of γ. In fact, we expect that $(d\sigma/d\varepsilon)_{\xi=0}$ starts to differ substantially from $d\sigma/d\varepsilon$ for ξ of the order of 1, i.e, for $\varepsilon/\gamma\simeq1$. It is thus more appropriate to plot Δ as a function of ε/γ. This is shown in Fig. 2 for the

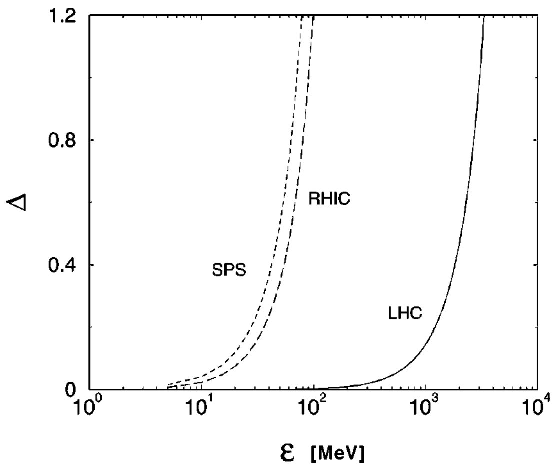

FIG. 1. Relative difference between the bound-free pair production spectrum calculated with the δ-function approximation and with the exact interaction, respectively [see Eq. (18)]. The comparison is done for SPS, RHIC, and LHC heavy-ion energies. The positron energies are given in MeV units in the laboratory system.

same laboratory energies as before. In this figure ε/γ is given in units of the electron mass. We see that all curves collapse into approximately a single one. The differences between the results for RHIC and LHC are imperceptible. These results show that the calculations with the δ-function interaction differ from the calculations with the exact potential for positron energies $\varepsilon \gtrsim 0.1\gamma mc^2$. For SPS and RHIC, this implies positron energies of the order of a few MeV, and for LHC energies of a few hundred MeV.

In the frame of reference of the nucleus where the electron is captured, the positrons move in a very forward direction, within an angle of the order of $1/\gamma$ along the projectile incident direction [4]. Thus, to study the impact parameter dependence of the production probabilities, we can safely use $\mathbf{p}=\mathbf{p}_z$. The differential probability for pair production is given by

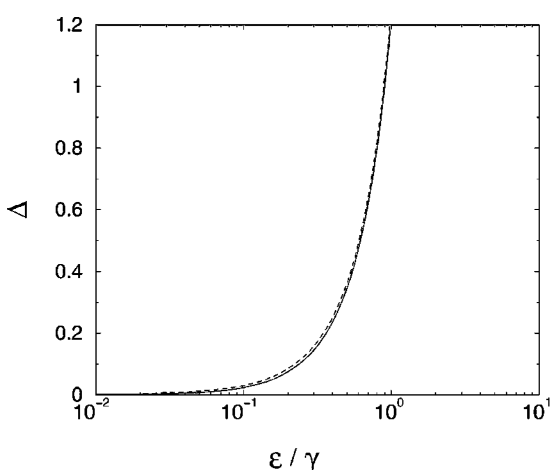

FIG. 2. Same as in Fig. 1, but as a function of the positron energies divided by the Lorentz γ factor. The positron energies are given in units of the electron rest mass.

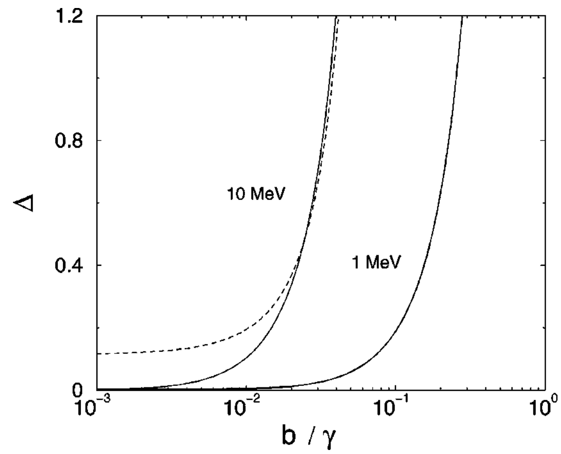

FIG. 3. Relative difference between the bound-free pair production probabilities calculated with the δ-function approximation and with the exact interaction, respectively [see Eq. (21)]. The comparison is done for SPS, RHIC, and LHC heavy-ion energies and for two different positron energies. The impact parameter divided by the Lorentz factor, b/γ, is given in units of the electron Compton wavelength $\hbar/mc$.

$$\frac{dP(b,\varepsilon)}{d\varepsilon d\Omega}=\frac{(Z\alpha)^2}{8\pi^5}(\varepsilon-1)p\left|\int d^2q_t e^{-i\mathbf{q}_t\mathbf{b}\cdot}\frac{f(\mathbf{q}_t)}{q_t^2+\xi^2}\right|^2. \tag{19}$$

Inserting the term inside brackets in Eq. (16) into the integral above, and neglecting terms of order $1/a_H^2$, one obtains the result

$$\frac{2\pi i}{\xi^2-\eta^2}\left\{\frac{1}{\xi^2-\eta^2}[\xi K_1(\xi b)-\eta K_1(\eta b)]-\frac{b}{2}K_0(\xi b)\right\}$$
$$-\frac{\pi i\xi}{\eta^4}K_0(\xi b), \tag{20}$$

where $\xi=\omega/\gamma v$ and $\eta=\omega/v-p$.

We define another function $\Delta(b,\varepsilon)$ to test the impact parameter dependence of the δ-function interaction,

$$\Delta(b,\varepsilon)=[(dP/d\varepsilon d\Omega)_{\xi=0}-dP/d\varepsilon d\Omega]/(dP/d\varepsilon d\Omega)_{\xi=0}. \tag{21}$$

In Fig. 3 we plot the function $\Delta(b,\varepsilon)$ for SPS, RHIC, and LHC heavy-ion energies, as a function of b/γ. For comparison we use two values of the positron energy: $\varepsilon=1$ and 10 MeV. For $\varepsilon=1$ MeV all curves agree, and one observes that the approximation is good up to impact parameters of the order of $b\simeq 0.1\gamma/\varepsilon$. This is confirmed by looking at the curves for $\varepsilon=10$ MeV. Then the agreement at SPS bombarding energies is not perfect, even for the smaller impact parameters. However, for RHIC and LHC energies, the results are basically equal. These results originate from the function given in Eq. (20), which drops sharply to zero at $\xi\simeq 1$, i.e., at positron energies of the order of $b\simeq\gamma/\varepsilon$. This is the so-called adiabatic limit. The electromagnetic field has photon energy components up to $1/t_{int}$, where t_{int} is the interaction time. This time is equals to $(b/\gamma)/v \simeq b/\gamma$ for relativistic collisions [3].

In conclusion, the present study has shown that δ-function interaction yields reasonable results as long as $\omega b/\gamma \lesssim 0.1$. As seen in Fig. 3, this amounts to $b \lesssim 0.1\gamma/\omega$. As observed in Ref. [3], the most effective impact parameters for this process are of the order of $b \simeq 1/m$. We also see in Figs. 1 and 2 that the differential cross sections $d\sigma/d\omega$ are well described up to energies of the order of 0.1γ.

For other situations, e.g., nuclear fragmentation due to the electromagnetic interaction in relativistic heavy ion collisions, the most effective impact parameter is given by $b \simeq R$, where $R \simeq 10$ fm. We thus expect that the δ-function interaction works well for $\varepsilon \lesssim 0.1\gamma$ MeV. Note that γ is the Lorentz factor in the frame of reference of one of the nuclei, i.e., $\gamma = 2\gamma_c^2 - 1$, where γ_c is the collider Lorentz factor. Thus γ is huge for RHIC and LHC energies, and the approximation works well for all energies of practical interest in nuclear fragmentation.

The basic idea of the δ-function interaction is that the electromagnetic field of a relativistic charge looks like a very thin pancake. Those processes which do not involve energy transfers that are too large, will not be sensitive to the spatial variation of the field. Then the δ-function is a good approximation. For typical energy transfers of the order of 10–100 MeV in nuclear fragmentation, the approximation works well for $b \lesssim (0.01–0.1)\gamma$ fm. To calculate total cross sections it is always necessary to account for those large impact parameters at which the δ-function approximation fails. Similar conclusions were drawn in a recent paper on projectile-electron loss in relativistic collisions with atomic targets [5].

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