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Bremsstrahlung radiation by a tunneling particle: A time-dependent description

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We study the bremsstrahlung radiation of a tunneling charged particle in a time-dependent picture. In particular, we treat the case of bremsstrahlung during α decay and show deviations of the numerical results from the semiclassical estimates. A standard assumption of a preformed α particle inside the well leads to sharp high-frequency lines in the bremsstrahlung emission. These lines correspond to ''quantum beats'' of the internal part of the wave function during tunneling arising from the interference of the neighboring resonances in the open well. [S0556-2813(99)50709-7]

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Recent experiments [1] have triggered great interest in the phenomena of bremsstrahlung during tunneling processes which was discussed from different theoretical viewpoints in Refs. [2–4]. This can shed light on basic and still controversial quantum-mechanical problems such as tunneling times [5], tunneling in a complex and nonstationary environment [6], preformation of a tunneling cluster in a many-body system [7], and so on. It seems that α decay offers a unique possibility to study these fundamental questions. The discussion of the bremsstrahlung radiation in α decay, initiated by a semiclassical calculation of Dyakonov and Gornyi [2] and continued by a quantum-mechanical calculation in perturbation theory of Papenbrock and Bertsch [3] and a more detailed comparison of quantum-mechanical calculations with classical and semiclassical results by Takigawa *et al.* [4], shows a complicated interference pattern arising from the contributions to bremsstrahlung from inner, under the barrier, and outer, in the classically allowed region, parts of the wave function. Currently the experimental data are not conclusive, and more exclusive experiments, with better statistics, are needed to give a clearer understanding of the phenomena.

The theoretical approaches of Refs. [3,4] assume a standard stationary description of quantum tunneling which is very successful for α-decay lifetime and probabilities. However, a time-dependent picture of a decay process [8,9] may be essential in understanding physics of the bremsstrahlung and similar processes, in particular, in obtaining the appropriate bremsstrahlung spectrum. This can be shown, for example, by looking into a case where there is no Coulomb acceleration after the tunneling. In this Rapid Communication we model the time evolution of a wave function during tunneling using again α decay as an example. We do not attempt to compare our results with experimental data. The reason is simple. The α decay time, e.g., the lifetime of ^{210}Po, is many orders of magnitude larger than typical times for an α-particle to traverse across the nucleus. This implies that a stable numerical solution of the Schrödinger equation, keeping track simultaneously of fast oscillations in the well (''escape attempts'') and extremely slow tunneling, is virtually impossible in this approach: for ^{210}Po it would require about 10^{30} time steps in the iteration process. Instead, we study the bremsstrahlung in high energy α decays, for which the decay time is treatable numerically. This allows us to pay attention to qualitatively new aspects of the bremsstrahlung in decay processes, and compare with stationary approaches. Although one could envisage other ways of solving the time-dependent Schrödinger equation which could possibly handle the real situation, this is beyond the scope of the present article, and we refer the reader to other methods, e.g., those discussed in Ref. [10].

For an α particle being accelerated from the turning point to infinity, the classical bremsstrahlung can be calculated analytically. Using well-known equations, see, for example [11,12], and integrating along the outward branch of the Rutherford trajectory in a head-on collision we get for the energy emitted by bremsstrahlung per frequency interval $d\omega$, in the long-wavelength approximation,

$$dE(\omega)=\frac{8\pi\omega^2}{3m^2c^3}Z_{\mathrm{eff}}^2e^2|p_r(\omega)|^2, \tag{1}$$

where

$$p_r(\omega)=\frac{ma}{2\pi}e^{-\pi\nu/2}K'_{i\nu}(\nu) \tag{2}$$

is the Fourier transform of the particle momentum, $m=m_N\cdot 4A/(A+4)$ is the reduced mass, and $Z_{\mathrm{eff}}=(2A-4Z)/(A+4)$ is the effective charge of the α particle + the daughter nucleus (A,Z) in the dipole approximation. In Eq. (1), $p_r=\mathbf{p}\cdot\hat{\mathbf{r}}$, $a=Ze^2/E_\alpha$, $\nu=E_\alpha a/\hbar v_0$, E_α is the energy of the α particle, and v_0 its asymptotic velocity. The functions $K_{i\nu}(x)$ are the modified Bessel functions of imaginary order, and $K'_{i\nu}(x)$ are their derivatives with respect to the argument. Dividing this equation by the photon energy $E_\gamma=\hbar\omega$, we get the differential probability per unit energy for the brems-

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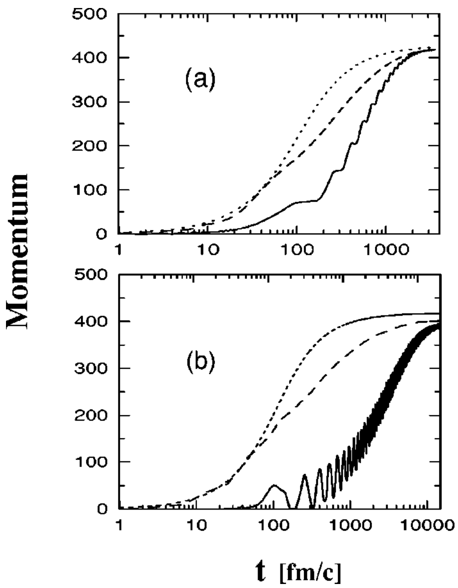

FIG. 1. Average momentum (in MeV/c) for α energies of 23.1 MeV (a) and 20 MeV (b). In (a) the dotted line is the classical momentum for an α particle running away from the closest approach distance. The dashed line is the average momentum calculated according to Eq. (4), but using only the part of the wave function outside the barrier, i.e., the integral in Eq. (4) extending from $R_{cl}= 2Ze^2/E_\alpha$ to infinity. The solid line represents the momentum calculated according to Eq. (4), but including the full space range of the wave function.

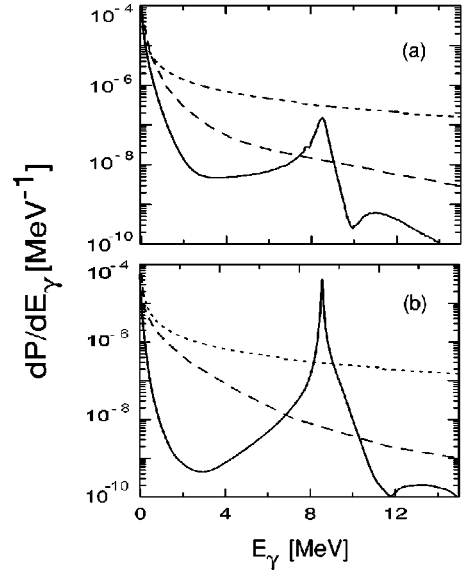

FIG. 2. Classical bremsstrahlung emission probability (dotted lines), quantum-mechanical spectrum (dashed lines) using only the part of the wave function outside the barrier, and using the full space wave function (solid lines). The upper part of the graph is for $E_\alpha = 23.1$ MeV, while the lower part is for $E_\alpha = 20$ MeV.

strahlung, i.e., $dP/dE_\gamma = (1/E_\gamma) dE(\omega)/dE_\gamma$. At low photon energies, we use the relation $\nu^2 e^{-\pi\nu}[K'_{i\nu}(\nu)]^2 \rightarrow 1$ to show that

$$\frac{dP}{dE_\gamma} = \frac{2}{3\pi}\frac{1}{E_\gamma} Z_{\text{eff}}^2 \alpha \left(\frac{v_0}{c}\right)^2. \tag{3}$$

We solve the time-dependent Schrödinger equation for the α particle in a potential well + Coulomb barrier, and calculate the radial momentum from

$$p_r(t) = \frac{\hbar}{i}\int dr u^*(r,t) \frac{\partial u(r,t)}{\partial r}, \tag{4}$$

where $u(r,t)$ is the radial part of the total wave function. A similar numerical study of the time evolution for the problem of the α decay was performed by Serot *et al.* [9].

The time-dependent Schrödinger equation is solved, starting with an s-wave wave function for an α particle confined in a radial well $V(r) = -V_0$ for $r < R_0$, and $V(r) = -V_0 + 2Ze^2/R_0$ for $r \geqslant R_0$. We set $R_0 = 8.75$ fm, realistic for the $\alpha + {}^{210}$Po case [3]. At $t=0$ we switch the potential to a spherical square well with depth $-V_0$ for $r < R_0$ and a Coulomb potential $2Ze^2/r$ for $r \geqslant R_0$ which triggers the tunneling. At a given time t the wave function is found by using the standard method of inversion of a tridiagonal matrix at each time step increment Δt (see, e.g., Ref. [15]). The space is discretized in steps of $\Delta x = 0.05$ fm, and the time step used is $\Delta t = mc(\Delta x)^2/\hbar$. In Fig. 1 we plot the average momentum for α energies of 23.1 MeV [Fig. 1(a)] and 20 MeV [Fig. 1(b)]. The α-energies are changed by keeping the number of nodes constant (in this case, seven nodes), and varying the depth of the potential from 14 to 18 MeV, respectively. The barrier height is 27.6 MeV. The dotted line is the classical momentum for an α particle running away from the closest approach distance. The dashed line is the average momentum calculated according to Eq. (4), but using only the part of the wave function outside the barrier, i.e., the integral in Eq. (4) extending from $R_{cl} = 2\,Ze^2/E_\alpha$ to infinity. In this case, the wave function entering (4) is normalized within this space range.

The resulting ''quantum-mechanical'' momentum at the later stage is closer to the classical one. However, the quantum-mechanical momentum increases slower than the classical one, due to the extended nature of the particle's wave function. As a consequence, one expects that the Fourier transform of the quantum momentum has its higher frequencies suppressed, as compared to the classical case. The solid line represents the momentum calculated according to Eq. (4), but including the full space range of the wave function. One observes a wiggling pattern associated with the interference of neighboring quasistationary states of the particle inside the nucleus. The leaking of the inner part of the wave function creates an effective oscillating dipole that emits radiation. The same leaking creates the perturbation mixing different resonance states. The Fourier transform of the particle momentum should therefore contain appreciable amplitudes associated with this motion, as observed in Fig. 2, solid line. Asymptotically, the momenta calculated in all different ways coincide at large times.

In Fig. 2 the dotted lines show the classical bremsstrahlung emission probability, the dashed lines show the ''quantum-mechanical'' momentum using only the part of the wave function outside the barrier, and the solid lines are for the full space range of the wave function, as a function of the photon momentum. The upper part of the graph is for $E_\alpha = 23.1$ MeV, while the lower part is for $E_\alpha = 20$ MeV. Since the quantum-mechanical momentum of the particle increases slower than the classical one, the spectrum at larger photon energies is suppressed in comparison with the classical one. Also, when one includes the whole wave function, the bremsstrahlung spectrum is even more suppressed at large photon energies. However, at very large photon energies ($E_\gamma \simeq 8.5$ MeV) the spectrum shows a peak, revealing the interference between the different components of the wave function in the well. The width ΔE of the peak is related to the lifetime of the quasistationary state. For $E_\alpha = 23$ MeV we find $\Delta E = 0.2$ MeV while for $E_\alpha = 20$ MeV the width is narrow, $\Delta E = 0.02$ MeV. These values agree with the conventional Gamow formula for the width of the quasistationary state of the α particle with the same energy and potential parameters.

In order to assess the relevance of the bremsstrahlung during tunneling and to shed light on the nature of the peaks in Fig. 2 we wil now consider an α particle confined within a three-dimensional square well with $V = -V_0$ for $r < R_0$, $V = U_0$ for $R_0 < r < R_1$, and $V = 0$, otherwise. We use the parameters $R_0 = 8.75$ fm, $R_1 - R_0 = 1$ fm, $V_0 = 14$ MeV, $U_0 = 27.6$ MeV, and $E_\alpha = 24.2$ MeV. For this problem a much simpler time-dependent solution can be found. The time-dependent wave function is obtained from the expansion

$$u(r,t) = \int dE\, a(E) e^{iEt/\hbar}\ u_E(r), \tag{5}$$

where $u_E(r)$ is the continuum (radial) wave function with energy E, normalized to $4\pi \int dr u_E^*(r) u_{E'}(r) = \delta(E - E')$ and

$$a(E) = 4\pi \int dr u_0(r)\ u_E(r), \tag{6}$$

where $u_0(r)$ is the radial wavefunction of the initial state. For a square well plus barrier $u_0(r)$, $u_E(r)$, and $a(E)$ are given analytically. The time-dependent wave function is obtained from Eq. (5) by a simple integration.

In Fig. 3(a) we plot the momentum of the particle for this system as a function of time. We observe a similar pattern as in Fig. 1(a), solid curve, but with stronger oscillations, due to the quantum beats. In Fig. 3(b) we show the corresponding bremsstrahlung spectrum. The resulting pattern is very similar to that displayed in Fig. 2. It is important to notice however that there is no Coulomb acceleration for this system. The lower part of the energy spectrum is due solely to the tunneling through the barrier, while the peak at higher energies is again due to the interference of the resonances during the tunneling process.

We have compared the lower part of the spectrum in Fig. 3(b) with the result of Dyakonov and Gornyi [2]. They have obtained the bremsstrahlung spectrum for a tunneling charge using perturbation theory and semiclassical wave functions for the initial and final state of the particle. We have done a similar calculation, but using the bound-state wave function

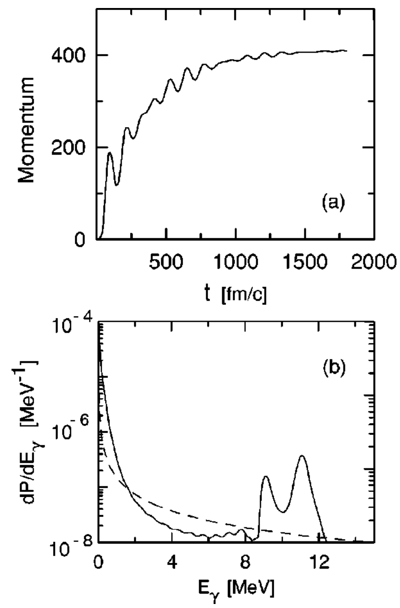

FIG. 3. (a) Momentum (in MeV/c) of a particle in a square well plus barrier as a function of time. (b) Corresponding bremsstrahlung spectrum.

$u_0(r)$ as the initial state. We obtained a bremsstrahlung spectrum for the soft part of the spectrum which is by far smaller than that displayed in Fig. 3(b). Moreover, the spectrum decays much faster than ours. This can be understood as follows. The soft part of the spectrum is due to the bremsstrahlung during tunneling. For a particle in a one-dimensional tunneling motion through a square barrier, Dyakonov and Gornyi's approach yields the spectrum given by

$$\frac{dP}{dE_\gamma} = \frac{4}{3\pi} \frac{U_0}{mc^2 E_\gamma} Z_{\rm eff}^2 \alpha\ \ \exp(-2k_1 d) \times \exp\left(-2\frac{E_\gamma d}{\hbar v_1}\right)\left[1 + \exp\left(-\frac{E_\gamma d}{\hbar v_1}\right)\right]^2, \tag{7}$$

where U_0 is the barrier height, $v_1 = \sqrt{2(U_0 - E_\alpha)/m}$ is the imaginary velocity of the particle during tunneling, and d is the barrier width.

This formula shows that the semiclassical bremsstrahlung spectrum of a tunneling particle varies as E_γ^{-1}, for low photon energies, and as $[\exp(-2E_\gamma d/\hbar v_1)]/E_\gamma$, for high photon energies. The slope parameter for high photon energies is given by $2d/\hbar v_1$ as displayed by the dashed line in Fig. 3(b). However, the spectrum obtained from the dynamical calculation, solid line in Fig. 3(b), has a smaller slope parameter. Additionally, the spectrum shows pronounced peaks at large photon energies. Figure 4, where we show the scattering phase shift for the system ''well + barrier,'' clarifies the origin of these peaks. The resonances at 15.7, 24.2, and 35.4 MeV correspond to (bound or virtual) levels in the well. The amplitudes of Eq. (6) are presented in Fig. 4(b) where the resonance peaks are also evident. The peaks at 9.1 MeV and 11.1 MeV, shown in Fig. 3(b), are due to interference of resonances at 15.7 MeV and 35.4 MeV with the initial state

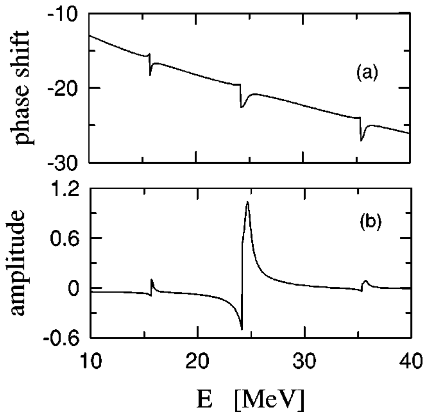

FIG. 4. (a) Phase shifts ($l=0$) for the $\alpha+{}^{210}$Po system, assuming a radial square well size of 8.75 fm + a square barrier of 1 fm located at the border of the well. (b) Amplitudes for the overlap of the initial state ($E=24.2$ MeV) of the closed well (barrier of infinitely large width) with neighboring states of the open well.

of energy 24.2 MeV. The quantum beats apparently correspond to the energy differences $(24.2-15.7=8.5)$ MeV and $(35.4-24.2=11.2)$ MeV, respectively. These values are close to the energies of the peaks appearing in Fig. 3(b).

The bremsstrahlung peaks associated with quantum beats are present in any dynamical tunneling process, since an initial localized state always has some overlap amplitude with neighboring states of the open well. The importance of these peaks, or, equivalently, of the admixture of neighboring resonances, decreases if the initial state is a very sharp resonance as in the case of the α decay of ^{210}Po studied in [1]. Until now it was poorly understood how an α particle was preformed inside a nucleus. The possible manifestations of virtually excited clusters in nuclei are predicted to be seen in knockout reactions induced by protons or electrons [7]. Studying bremsstrahlung radiation in the tunneling process can provide additional information. Indeed, the initial wave function must be of a localized nature, thus having a nonzero amplitude for carrying a part of the wave function of an adjacent resonance. For high-lying states, as shown above, this leads to pronounced peaks in the bremsstrahlung spectrum. The observation of those peaks would be valuable for inferring the content of the initial wave function of a preformed α particle (or fission product).

The coupling to the radiation field can also influence the tunneling process. Such effects require a fully quantum-mechanical approach which can be formulated in spirit of the classical work by Pauli and Fierz [13]. It includes important feedback effects which renormalize the particle trajectory by the coupling to the accompanying radiation field [14]. The exact solution of these equations is crucial in the case where the photon energy is comparable to the particle energy. For example, in the decay of very low energy α's this coupling may reduce the yield of high energy photons. The reason is that when a particle emits a photon before tunneling, it looses energy and this leads to a reduction of its barrier tunneling probability. The exact results might be sensitive to the shape of the barrier. We hope to come to this formulation of the problem elsewhere.

In conclusion, we have obtained the bremsstrahlung spectrum of a tunneling particle (an α particle in a nucleus) by directly solving the time-dependent Schrödinger equation. As expected, we have found that there are large deviations from the classical bremsstrahlung spectrum. We have also demonstrated that aproaches based on perturbation theory miss an important piece of information, namely, the time-dependent modification of the particle wave function in the well during the decay time. This leads to substantial emission of photons with frequencies close to those of quantum beats between neighboring resonances. This effect should be relevant in radiation emitted during α decay in nuclei. In a more general case, the time dependence of the wave function of a tunneling particle seems to deviate substantially from the spectrum calculated by using perturbation theory with semiclassical wave functions. More experimental data on bremsstrahlung radiation by a tunneling particle would be very welcome in learning more about preformation states and dynamics of quantum beats.

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