Elastic Electron Scattering off Exotic Nuclei

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I. INTRODUCTION

The use of radioactive nuclear beams produced by fragmentation in high-energy heavy-ion reactions lead to the discovery of halo nuclei, such as ^{11}Li, about 20 years ago [1]. Nowadays a huge number of β-unstable nuclei far from stability are being studied thanks to further technical improvements. Unstable nuclei far from stability are known to play an important role in nucleosynthesis. Detailed studies of the structure and their reactions will have unprecedented impact on astrophysics [2].

The first experiments with unstable nuclear beams aimed at determining nuclear sizes by measuring the interaction cross section in high energy collisions [1]. Successive use of this technique has yielded nuclear size data over a wide range of isotopes. Other techniques, e.g. isotope-shift measurements, have allowed to extract the charge size. The growth of a neutron skin with the neutron number in several isotopes have been deduced from nuclear- and charge-size data [3].

Undoubtedly, electron scattering off nuclei would provide the most direct information about charge distribution, which is closely related to the spatial distribution of protons [4]. A technical proposal for an electron-heavy-ion collider has been incorporated in the GSI/Germany physics program [5]. A similar program exists for the RIKEN/Japan facility [6]. In both cases the main purpose is to study the structure of nuclei far from the stability line. The advantages of using electrons in the investigation of the nuclear structure are mainly related to the fact that the electron-nucleus interaction is relatively weak. For this reason multiple scattering effects are usually neglected and the scattering process is described in terms of perturbation theory. Since the reaction mechanism in perturbation theory is well under control the connection between the cross section and quantities such as charge distributions, transition densities, response functions etc., is well understood [7].

Under the impulse approximation, or plane wave Born approximation, the charge form factor can be determined from the differential cross section of elastic electron scattering. Since the charge distribution, $\rho_{ch}(r)$, is obtained from the charge form factor by a Fourier transformation, one can experimentally determine $\rho_{ch}(r)$ by differential cross-section measurements covering a wide range of momentum transfer q. One can obtain information on the size and diffuseness when the charge form factor is measured at least up to the first maximum. To do this within a reasonable measuring time of one week, a luminosity larger than 10^{26} cm^{-2}s^{-1} is required, for example for the ^{132}Sn isotope [5].

On the theoretical side the difference between the proton and neutron distributions can be obtained in the framework of Hartree-Fock (HF) method (see for example [8]) or Hartree-Fock-Bogoliubov (HFB) method (see for example [9, 10]). As a rule of thumb, a theoretical calculation of the nuclear density is considered good when it reproduces the data on elastic electron scattering. But some details of the theoretical densities might not be accessible in the experiments, due to poor resolution or limited experimental reach of the momentum transfer q.

II. ELASTIC ELECTRON SCATTERING

In the plane wave Born approximation (PWBA) the relation between the charge density and the cross section is given by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm PWBA} = \frac{\sigma_M}{1+(2E/M_A)\sin^2(\theta/2)}\,|F_{ch}(q)|^2, \tag{1}$$

where $\sigma_M = (Z^2\alpha^2/4E^2)\cos^2(\theta/2)\sin^{-4}(\theta/2)$ is the Mott cross section, the term in the denominator is a recoil correction, E is the electron total energy, M_A is the mass of the nucleus and θ is the scattering angle.

The charge form factor $F_{ch}(q)$ for a spherical mass distribution is given by

$$F_{ch}(q) = 4\pi\int_0^\infty dr\; r^2 j_0(qr)\,\rho_{ch}(r)\,, \tag{2}$$

where $q = 2k\sin(\theta/2)$ is the momentum transfer, $\hbar k$ is the electron momentum, and $E = \sqrt{\hbar^2k^2c^2 + m_e^2c^4}$. The low momentum expansion of eq. 2 yields the leading terms $F_{ch}(q)/Z = 1 - q^2\langle r_{ch}^2\rangle/6 + \cdots$. Thus, a measurement at low momentum transfer yields a direct assessment of the mean square radius of the charge distribution, $\langle r_{ch}^2\rangle^{1/2}$. However, as more details of the charge distribution is probed more terms of this series are needed and, for a precise description of it, the form factor dependence for large momenta q is needed.

A theoretical calculation of the charge density entering eq. 2 can be obtained in many ways. Let $\rho_p(\mathbf{r})$ and $\rho_n(\mathbf{r})$ denote the point distributions of the protons and the neutrons, respectively, as calculated, e.g. from single-particle wavefunctions obtained from an average one-body potential well, the latter in general being different for protons and neutrons. If $f_{Ep}(\mathbf{r})$ and $f_{En}(\mathbf{r})$ are the spatial charge distributions of the proton and the neutron in the non-relativistic limit, then the charge distribution of the nucleus is given by

$$\rho_{ch}(\mathbf{r}) = \int \rho_p(\mathbf{r}')\;\; f_{Ep}(\mathbf{r}-\mathbf{r}')\;\; d^3r' + \int \rho_n(\mathbf{r}')\;\; f_{En}(\mathbf{r}-\mathbf{r}')\;\; d^3r'. \tag{3}$$

The second term on the right-hand side of eq. 3 plays an important role in the interpretation of the charge distribution of some nuclear isotopes. For example, the half-density charge radius increases 2% from ^{40}Ca to ^{48}Ca, whereas the surface thickness decreases by 10% with the result that there is more charge in the surface region of ^{40}Ca than of ^{48}Ca [11]. This also results that the rms charge radius of ^{48}Ca is slightly smaller than that of ^{40}Ca. The reason for this anomaly is that the added $f_{7/2}$ neutrons contribute negatively to the charge distribution in the surface and more than compensate for the increase in the rms radius of the proton distribution.

For the proton the charge density $f_{Ep}(r)$ in eq. 3 is taken as an exponential function, what corresponds to a form factor $G_{Ep}(q) \equiv \widetilde{f}_{Ep}(q) = (1+q^2/\Lambda^2)^{-1}$. For the neutron a good parametrization is $G_{En}(q) = -\mu_n\tau G_{Ep}(q)/(1+p\tau)$, where μ_n is the neutron magnetic dipole moment and $\tau = q^2/4m_N$. One can use $\Lambda^2 = 0.71$ fm^{-2} (corresponding to a proton rms radius of 0.87 fm) and $p = 5.6$, which Galster et al. [12] have shown to reproduce electron-nucleon scattering data.

Eqs. 1-3 are based on the first Born approximation. They give good results for light nuclei (e.g. ^{12}C) and high-energy electrons. For large-Z nuclei the agreement with experiments is only of a qualitative nature. The effects of distortion of the electron waves have been

studied by many authors (see, e.g. ref. [13, 14, 15]). More important than the change in the normalization of the cross section is the displacement of the minima. It is well known that a very simple modification can be done in the PWBA equation that reproduces the shift of the minima to lower q's. One replaces the momentum transfer q in the form factor of eq. 1 with the effective momentum transfer $q_{eff} = q\,(1 + 3Ze^2/2R_{ch}E)$, where E is the electron energy and $R \simeq 1.2\ A^{-1/3}$ fm. This is because a measurement at momentum transfer q probes in fact $|F(q)|^2$ at $q = q_{eff}$ due to the attraction the electrons feel by the positive charge of the nucleus. This expression for q_{eff} assumes a homogeneous distribution of charge within a sphere of radius R.

A realistic description of the elastic electron scattering cross section requires the full solution of the Dirac equation. The Dirac equation for the elastic scattering from a charge distribution can be found in standard textbooks, e.g. [16]. Numerous DWBA codes based on Dirac distorted waves have been developed and are public.

III. SKINS AND HALOS

A. Neutron Skins

Appreciable differences between neutron and proton radii are expected [17] to characterize the nuclei at the border of the stability line. The liquid drop formula expresses the binding energy of a nucleus with N neutrons and Z protons as a sum of bulk, surface, symmetry and Coulomb energies $E/A = -a_V A + a_S A^{2/3} + S(N-Z)^2/A + a_C Z^2/A^{1/3} \pm a_p A^{-1/2}$,where a_V, a_S, a_p, S and a_C are parameters fitted to the experimental data of binding energy of nuclei. This equation does not distinguish between surface (S) and volume (V) symmetry energies. As shown in ref. [18], this can be achieved by partitioning the particle asymmetry as $N-Z = N_S - Z_S + N_V - Z_V$. The total symmetry energy S then takes on the form $S = S_V(N_V - Z_V)^2/A + S_S(N_S - Z_S)^2/A^{2/3}$. Minimizing under fixed $N-Z$ leads to an improved liquid drop formula [18] with the term $S_V(N-Z)^2/A$ replaced by $S_V(N-Z)^2/A[1 + (S_V/S_S)\,A^{-1/3}]$. The same approach also yields a relation between the neutron skin $R_{np} = R_n - R_p$, and S_S, S_V, namely [18]$R_n - R_p/R = (A/6NZ)[N - Z - (a_C/12S_V)\,ZA^{2/3}]/[1 + (S_S/S_V)\,A^{1/3}]$, where $R = (R_n + R_p)/2$.

Here the Coulomb contribution is essential; e.g. for $N = Z$ the neutron skin R_{np} is negative due to the Coulomb repulsion of the protons. A wide variation of values of S_V and S_S can be found in the literature. These values have been obtained by comparing the above predictions for energy and neutron skin to theoretical calculations of nuclear densities and experimental data on other observables [18, 19]. For heavy nuclei, with $A \gg 1$, $NZ \simeq A^2/2$, and using $a_C = 0.69$ MeV, $S_V = 28$ MeV and $R = 1.2A^{1/3}$ fm, one gets

$$R_{np} = R_n - R_p \simeq 0.4 \left(\frac{S_V}{S_S}\right) \left(\delta - 2.05 \times 10^{-3} Z A^{-1/3}\right) \text{ fm}, \tag{4}$$

where $\delta = (N-Z)/A$ is the asymmetry parameter. If one assumes that the central densities for neutrons and protons are roughly the same and that they are both described by a uniform distribution with sharp-cutoff radii, R_n and R_p, one finds $R_{np} \simeq 0.8A^{1/3}\delta$ fm. Of course, the sharp sphere model is too simple.

On the experimental front, a study of antiprotonic atoms published in reference [20] obtained the following fitted formula for the neutron skin of stable nuclei in terms of the

root mean square (rms) radii of protons and neutrons

$$\Delta r_{np} = \left\langle r_n^2 \right\rangle^{1/2} - \left\langle r_p^2 \right\rangle^{1/2} = (-0.04 \pm 0.03) + (1.01 \pm 0.15)\,\delta \text{ fm}. \quad (5)$$

The relation of the mean square radii with the half-density radii is given by $\langle r_n^2 \rangle = 3R_n^2/5 + 7\pi^2 a_n^2/5$, where a_n is the diffuseness parameter. For heavy nuclei, assuming $a_n = a_p \ll R_n, R_p$, one gets the same linear dependence on the asymmetry parameter as in eq. 4.

Eq. 5 can be used as the starting point for accessing the dependence of electron scattering on the neutron skin of heavy nuclei. Applying it to calcium isotopes as an example, one obtains that the neutron skin varies from -0.15 fm for ^{35}Ca (proton-rich with negative neutron skin) to 0.25 fm for ^{53}Ca. A negative neutron skin obviously means an excess of protons at the surface.

For heavy nuclei the charge and neutron distributions can be described by a Fermi distribution. The diffuseness is usually much smaller than the half-density radius, $a_{p,n} \ll R_{p,n}$. The neutron skin is then given by $R_{np} = R_n - R_p \simeq \sqrt{5/3}\ \Delta r_{np}$. One can assume further that the nuclear charge radius is given by $R_p = 1.2A^{1/3}$ fm $- R_{np}/2$ with Δr_{np} given by eq. 5. The first and second minima of the form factors occur at $q_1 = 4.49/R_p$ and $q_2 = 7.73/R_p$, respectively, corresponding to the zeroes of the transcendental equation $\tan(qR_p) = qR_p$.

The linear dependence of R_p with the neutron skin (and with the asymmetry parameter δ), also imply a linear dependence of the position of the minima,

$$q_1 \simeq \frac{3.74}{A^{1/3}} \left[1 - 0.535\frac{(N-Z)}{A^{4/3}}\right]^{-1} \text{ fm}^{-1}, \qquad \text{and} \qquad q_2 = 1.72\ q_1\ . \quad (6)$$

For ^{100}Sn the first minimum is expected to occur at $q_1 = 0.806$ fm$^{-1} = 159$ MeV/c, while for ^{132}Sn it occurs at $q_1 = 0.754$ fm$^{-1} = 149$ MeV/c.

The variation of q_1 with the neutron skin of neighboring isotopes, $\Delta q_1 \simeq 2/A^{8/3}$ fm^{-1}, is too small to be measured accurately. The first minimum, q_1, changes from 220 MeV/c for ^{35}Ca to 204 MeV/c for ^{53}Ca, an approximate 7%, which is certainly within the experimental resolution. Of course, sudden changes of the neutron skin with δ might happen due to shell closures, pairing, and other microscopic effects.

To be more specific, let us assume that a reasonable goal is to obtain accurate results for the charge radius $\left\langle r_p^2 \right\rangle^{1/2}$, so that $\delta \left\langle r_p^2 \right\rangle^{1/2} < 0.05$ fm. This implies that the measurement of q_1 has to be such that $(\Delta q_1/q_1) < q_1\,[\text{fm}^{-1}]$ %, with q_1 in units of fm^{-1} and the right-hand side of the inequality yielding the percent value. For ^{53}Ca, $q_1 = 1.11$ fm^{-1} meaning that the experimental resolution on the value of q_1 has to be within 1% if $\delta \left\langle r_p^2 \right\rangle^{1/2} < 0.05$ fm is a required precision. Of course, the ultimate test of a given theoretical model will be a good reproduction of the measured data, below and beyond the first minimum.

B. Neutron halos

Elastic electron scattering will be very important to determine charge distributions in proton-rich nuclei. This will complement the basic information on the charge distribution in, e.g. ^{8}B, obtained in nucleon knockout reactions [21]. For light nuclei composed by a core nucleus and an extended distribution of halo nucleons, the nuclear matter form factor can be fitted with the simple expression $F(q) = (1-g)\exp(-q^2a_1^2/4) + g/(1+a_2^2q^2)$, with the density normalized to one. g is the fraction of nucleons in the halo. In this expression

the first term follows from the assumption that the core is described by a Gaussian and the halo nucleons by an Yukawa distribution. Taking ^{11}Li as an example, the following set of parameters can be used $g = 0.18$, $a_1 = 2.0$ fm and $a_2 = 6.5$ fm. Although only few nucleons are in the halo they change dramatically the appearance of the squared form factor. Even when the individual contribution of the halo nucleons is small and barely visible in a linear plot of the matter distribution, it is very important for the form factor of the total matter distribution. It is responsible for the narrow peak which develops at low momentum transfers. This signature of the halo was indeed the motivation for the early experiments with radioactive beams. The narrow peak was observed in momentum distributions following knockout reactions [1].

Elastic electron scattering will not be sensitive to the narrow peak of $|F(q)|^2$ at small momentum q, but to the form factor of the charge distribution, $|F_{ch}(q)|^2$. The determination of this form factor will tell us if the core has been appreciably modified due to the presence of the halo nucleons.

In order to explain the spin, parities, separation energies and size of exotic nuclei consistently a microscopic calculation is needed. One possibility is to resort to a Hartree-Fock (HF) calculation. Unfortunately, the HF theory cannot provide the predictions for the separation energies within the required accuracy of hundred keV. I have used a simple and tractable HF method [22] to generate synthetic data for the charge-distribution of ^{11}Li. Details of this method is described in ref. [23]. Assuming spherical symmetry, the equation for the Skyrme interaction can be written as

$$\left[-\nabla\frac{\hbar^2}{2m^*(r)}\nabla + V(\mathbf{r})\right]\psi_i(\mathbf{r}) = \epsilon_i\psi_i(\mathbf{r}) \tag{7}$$

where $m^*(r)$ is the effective mass. The potential $V(\mathbf{r})$ has a central, a spin-orbit and a Coulomb term. The central potential is multiplied by a constant factor f only for the last neutron configuration:

$$V_{\rm central}(r) = fV_{HF}(r), \begin{cases} f \neq 1 & \text{for last neutron configuration} \\ f = 1 & \text{otherwise.} \end{cases} \tag{8}$$

As the effective interaction, a parameter set of the density dependent Skyrme force, so called BKN interaction [24], is adopted. The parameter set of BKN interaction has the effective mass m*/m =1 and gives realistic single particle energies near the Fermi surface in light nuclei. The original BKN force has no spin-orbit interaction. In the calculations, the spin-orbit term was introduced in the interaction so that the single-particle energy of the last neutron orbit becomes close to the experimental separation energy. In this way, the asymptotic form of the loosely-bound wave function becomes realistic in the neutron-rich nucleus. In order to obtain the nuclear sizes, the rms radii of the occupied nucleon orbits are multiplied by the shell model occupation probabilities, which are also obtained in the calculations.

The elastic form factor for the matter distribution obtained in the HF calculations are very close to the ones calculated by the empirical formula given at the beginning of this section. The lack of minima, and of secondary peaks (as in the empirical formula), make it difficult to extract from $|F_{ch}(q)|^2$ more detailed information on the charge-density profile. For example, in the case of ^{6}Li a good fit to experimental data was obtained with [25] $|F_{ch}(q)|^2 \propto \exp(-a^2q^2) - Cq^2\exp(-b^2q^2)$, with $a = 0.933$ fm, $b = 1.3$ fm, and $C = 0.205$. However, the data cannot be fitted by using a model in which the nucleons move in a single-particle potential.

C. Proton Halos

I will consider ^{8}B as a prototype of proton halo nucleus. This nucleus is perhaps the most likely candidate for having a proton halo structure, as its last proton has a binding energy of only 137 keV. The charge density for this nucleus can be calculated in the framework of the Skyrme HF model. I use here the results obtained in ref. [26], where axially symmetric HF equations were used with SLy4 [27] Skyrme interaction which has been constructed by fitting the experimental data on radii and binding energies of symmetric and neutron-rich nuclei. Pairing correlations among nucleons have been treated within the BCS pairing method. The form factor squared for the charge density in ^{8}B is calculated.

The width of the charge form factor squared is $\Delta_{ch} = 0.505$ fm^{-1} = 99.6 MeV/c. The corresponding values for the neutron and the total matter distributions are, respectively, $\Delta_n = 0.512$ fm^{-1} = 101 MeV/c and $\Delta_{tot} = 0.545$ fm^{-1} = 108 MeV/c. This amounts to approximately 10% differences of matter and charge form factors in ^{8}B.

The proton halo in ^{8}B is mainly due to the unpaired proton in the $p_{3/2}$ orbit. It is clear that for such a narrow halo the size of the nucleon also matters. A slice of the nucleon is included in a thin spherical shell of radius r and thickness dr from the center of the nucleus. If the position of the nucleon is given by R, the part of the proton charge included in the spherical shell is given by

$$d\rho_{ch} = 2\pi r^2 dr \int_0^\pi d\theta \ \rho_p(\mathbf{x}) \sin\theta, \tag{9}$$

where $\rho_p(\mathbf{x})$ is the charge distribution inside a proton at a distance $\mathbf{x}$ from its center. The coordinates are related by $x^2 = r^2 + R^2 - 2rR\cos\theta$. The contribution to the nuclear charge distribution from a single-proton in this spherical shell is thus given by $\mathcal{N}_p(R,r) = d\rho_{ch}/4\pi r^2 dr$.

Assuming that the charge distribution of the proton is described either by a Gaussian or a Yukawa form, the integral in eq. 9 can be performed analytically, yielding

$$\mathcal{N}_p^{(G)}(R,r) = \frac{1}{4\pi^{1/2} arR} \begin{cases} \frac{1}{\pi}\left\{\exp\left[-\frac{(R-r)^2}{a^2}\right] - \exp\left[-\frac{(R+r)^2}{a^2}\right]\right\}, & \text{for a Gaussian dist.} \\ \frac{1}{2}\left\{\exp\left[-\frac{|R-r|}{a}\right] - \exp\left[-\frac{|R+r|}{a}\right]\right\}, & \text{for a Yukawa dist.,} \end{cases} \tag{10}$$

where a is the proton radius parameter.

The charge distribution at the surface of a heavy proton-rich nucleus, $\delta\rho_{ch}(r)$, may be described as a pile-up of protons forming a skin. Let n_i be the number of protons in the skin and R_i their distance to the center of the nucleus. One gets

$$\delta\rho_{ch}(r) = \sum_i n_i \mathcal{N}_p(r, R_i). \tag{11}$$

Assuming R_i to be constant, equal to the nuclear charge radius R, and using eq. 10 it is evident that while the density at the surface increases, its size R and width a, remain unaltered. The form factor associated with this charge distribution is given by

$$\delta F_{ch}(q) = \frac{4\pi}{q} \sum_i n_i \int_0^\infty dr \ r \ \mathcal{N}_p(r, R_i) \ \sin(qr) = \exp\left(-qa^2\right) \sum_i n_i \frac{\sin(qR_i)}{qR_i}, \tag{12}$$

where the last result is for the Gaussian distribution. An analytical expression can also be obtained for the Yukawa distribution. For $R_i = R$, expression 12 shows that the increase of density in the skin does not change the shape of the form factor, or of the cross section, but just its normalization. The fall down of the cross section is determined by a alone, and not by n_i. If the charge of additional protons is distributed homogeneously across the nucleus including the skin, the normalized form factor will not change, except for a small change in R.

For a proton halo nucleus it is more appropriate to replace $\sum_i n_i \rightarrow 4\pi \int dR\ R^2\ \mathcal{N}_p\left(r,R\right)\delta\rho_{ch}\left(R\right)$, where $\delta\rho_{ch}\left(R\right)$ is the density change created by the extended wavefunction of the halo protons. One then recast eq. 12 in the form

$$\delta F_{ch}\left(q\right) = \frac{4\pi}{q}\exp\left(-qa^2\right)\int_0^\infty dR\ R\ \delta\rho_{ch}\left(R\right)\ \sin\left(qR\right). \tag{13}$$

The shape of the form factor is here dependent not only on the proton size but also on the details of the halo density distribution. For ^{8}B, the halo size is determined by the valence proton in a $p_{3/2}$ orbit. The density $\delta\rho_{ch}\left(R\right)$ due this proton can be calculated with a Woods-Saxon model. Using the same potential parameters as in ref. [28] I compare the form factor $\left|\delta F_{ch}\left(q\right)\right|^2$ to the charge form factor $\left|F_{ch}\left(q\right)\right|^2$. The halo is found to contribute to a narrow form factor. However, in contrast to the neutron halo case, the charge form factor of ^{8}B does not show a pronounced influence of the halo charge distribution.

The rms radius of the charge distribution can be calculated from $\langle r_{ch}^2\rangle = -6\left[dF_{ch}/d\left(q^2\right)\right]_{q^2=0}$. Applying this relationship to the ^{8}B charge form factor we get $\langle r_{ch}^2\rangle^{1/2} = 2.82$ fm which is close to the experimental value $\langle r_{ch}^2\rangle_{\rm exp}^{1/2} = 2.76 \pm 0.08$ fm. The shape of the charge form factor can also be described by a Gaussian distribution with radius parameter $a = 2.30$ fm. In contrast to the case of ^{11}Li, the proton halo in ^{8}B does not seem to build up a two-Gaussian shaped form factor. This observation also seems to be compatible with the momentum distributions of ^{7}Be fragments in knockout reactions using ^{8}B projectiles in high energy collisions [21]. Electron scattering experiments will help to further elucidate this property of proton halos.

IV. INVERSE SCATTERING PROBLEM

In PWBA the inverse scattering problem can be easily solved. It is possible to extract the form factor from the cross section and then, with an inversion of the Fourier transform, to get the charge density distribution

$$\rho_{ch}\left(r\right) = \frac{1}{2\pi^2}\int_0^\infty dq\ q^2 j_0\left(qr\right)F_{ch}\left(q\right). \tag{14}$$

The PWBA approximation can be justified only for light nuclei (e.g. ^{12}C) in the region far from the diffraction zeros. For higher Z values the agreement with experiment is only of a qualitative nature.

It is very common in the literature to use a theoretical model for $\rho_{ch}\left(r\right)$, e.g. the HF calculations discussed in the previous sections and compare the calculated $F(q)$ with experimental data. When the fit is "reasonable" (usually guided by the eye) the model is considered a good one. However, whereas the theoretical $\rho_{ch}\left(r\right)$ can contain useful information about the central part of the density (e.g. bubble-like nuclei, with a depressed central

density), an excellent fit to the available experimental data does not necessarily mean that the data is sensitive to those details. The obvious reason is that short distances are probed by larger values of q. Experimental data from electron-ion colliders will suffer from limited accuracy at large values of q, possibly beyond $q = 1$ fm^{-1}.

In order to obtain an unbiased "experimental" $\rho_{ch}(r)$ one usually assumes that the density is expanded as $\rho_{ch}(r) = \sum_{n=1}^{\infty} a_n f_n(r)$, where the basis functions $f_n(r)$ are drawn from any convenient complete set and the expansion coefficients a_n are adjusted to reproduce the differential elastic cross section. The corresponding Fourier transform then takes the form

$$\widetilde{\rho}(q) \equiv F_{ch}(q) = \sum_{n=1}^{\infty} a_n \widetilde{f}_n(q), \qquad \widetilde{f}_n(q) = 4\pi \int_0^{\infty} dr\ r^2 j_0(qr) f_n(r). \tag{15}$$

Evidently the sum in n has to be truncated and this produces an error in the determination of the charge density distribution. Another problem is that, as shown by eq. 14, the solution of the inverse scattering problem requires an accurate determination of the cross section up to large momentum transfers. Electron scattering experiments in electron-ion colliders will be performed within a limited range of q and this will produce an uncertainty in the determination of the charge density distribution.

Two bases have been found useful [29] in the analysis of electron or proton scattering data. The present discussion is limited to spherical nuclei, but generalizations to deformed nuclei can be done. The Fourier-Bessel (FB) expansion (i.e. with f_n taken as spherical Bessel functions) is useful because of the orthogonality relation between spherical Bessel functions

$$\int_0^{R_{\max}} dr\ r^2 j_l(q_n r) j_l(q_m r) = \frac{1}{2} R_c^3 \left[j_{l+1}(q_n R_{\max})\right]^2 \delta_{nm}, \tag{16}$$

where the q_n are defined such as $j_l(q_n R_{\max}) = 0$. The FB basis implies that the charge density $\rho_{ch}(r)$ should be zero for values of r larger than $R_{\max}$. For example, the basis can be defined as follows

$$f_n(r) = j_0(q_n r)\,\Theta(R_{\max} - r), \qquad \widetilde{f}_n(q) = \frac{4\pi (-1)^n R_{\max}}{q^2 - q_n^2} j_0(qR_{\max}), \tag{17}$$

where Θ is the step function, $R_{\max}$ is the expansion radius and $q_n = n\pi/R_{\max}$.

In principle it is possible to obtain the a_n coefficients measuring directly the cross section at the q_n momentum transfer. If the form factor (2) is known at q_n, the coefficients a_n can be obtained inserting (16) and (17) in the definition (2) of the form factor, leading to

$$a_n = \frac{F_{ch}(q_n)}{2\pi R_{\max}^3 \left[j_1(q_n R_{\max})\right]^2}. \tag{18}$$

In general the cross sections are measured at q values different from q_n. Using the expansion (17) of the charge density one finds for the form factor the relation

$$F_{ch}(q) = \frac{4\pi}{q} \sum_n a_n \frac{(-1)^n}{q^2 - q_n^2} \sin(qR_{\max}). \tag{19}$$

By fitting the experimental $F_{ch}(q)$ one obtains the a_n parameters and reconstruct the nuclear charge density. Not all a_n's are needed. Since the integral of the density, or $F(q = 0)$, is

fixed to the charge number there is one less degree of freedom. Also, densities tend to zero at large r. Thus another condition can be used, e.g. that the derivative of the density is zero at $R_{\max}$. Thus, when one talks about n expansion coefficients one means in fact that only $n-2$ coefficients need to be used in eq. 19. For experiments performed up to $q_{\max}$ the number of expansion coefficients needed for the fit is determined by $n_{\max} \simeq q_{\max} R_{\max}/\pi$.

A disadvantage of the FB expansion is that a relatively large number of terms is often needed to accurately represent a typical confined density, e.g. for light nuclei. One can use other expansion functions which are invoke less number of expansion parameters, e.g. the Laguerre-Gauss (LG) expansion,

$$f_n(r) = e^{-\alpha^2} L_n^{1/2}\left(2\alpha^2\right), \qquad \text{and} \qquad \widetilde{f}_n(q) = 4\pi^{3/2}\beta^3 (-1)^n e^{-\gamma^2} L_n^{1/2}\left(2\gamma^2\right),$$

where $\alpha = r/\beta$, $\gamma = q\beta/2$, and L_n^p is the generalized Laguerre polynomial. Another possibility is to use an expansion on Hermite (H) polynomials. In both cases, the number of terms needed to provide a reasonable approximation to the density can be minimized by choosing β in accordance with the natural radial scale. For light nuclei $\beta = 1-2$ fm can be chosen, consistent with the parametrization of their densities. Then the magnitude of a_n decreases rapidly with n, but the quality of the fit and the shape of the density are actually independent of β over a wide range.

For real data, the expansion coefficients a_n are obtained by minimizing

$$\chi^2 = \sum_i \left(\frac{y_i - y(q_i, a_n)}{\sigma_i}\right)^2,$$

where $y(q_i, a_n)$ is the fitted value of the cross section (form factor) with a set of coefficients a_n and y_i are the experimental data at momentum q_i with uncertainty σ_i.

Increasing the number of coefficients does not improve the quality of the fit. It only produces more oscillations of the density. The reason is that terms with larger n's are only needed to reproduce the data at larger values of momentum transfer. The fit to the data for $q < q_{\max}$ is not affected but the presence of these new terms introduces oscillations in the charge distribution. A possible fix to this problem is to include pseudodata in addition to experimental data. This method is well known in the literature [29]. The pseudodata are used to enforce constraints and to estimate the incompleteness error associated with the limitation of experimental data to a finite range of momentum transfer.

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