Notes and Tones

• Musical instruments cover useful range of 27 to 4200 Hz.

• **Ear:** pitch discrimination of 0.03 semitones → 30 distinguishable pitches in one semitone. (much more than needed!). (one semitone = 1/12 of an octave)

• Musicians select discrete frequencies in an array: **SCALE**

• One of the frequencies = **NOTE**

• Note is also a symbol in a musical staff, or refers to a key on a piano, etc.

• Note is sometimes synonymous to **TON**
Scale and Temperament

Scale - A succession of notes in ascending order (e.g., Pythagorean, just, meantone, equal temperament).

Tuning - Adjustment of pitch to correspond to an accepted norm.

Temperament - A system of tuning in which intervals deviate from acoustically pure (Pythagorean).

Intonation - Degree of accuracy with which pitches are produced.
Pythagoras and the **monochord**

**Ancient Greeks** - Aristotle and his followers - discovered using a **Monochord** that certain combinations of sounds with **rational number** \((n/m)\) frequency ratios were pleasing to the human ear.

\[
f \propto \frac{1}{L} \quad \Rightarrow \quad \frac{f_1}{f_2} \propto \frac{L_2}{L_1}
\]
Jump few centuries: Piano keyboard

Do   Re  Me   Fa  So   La   Ti   Do

C_1  C_2  C_3  C_4  C_5  C_6  C_7  C_8

C_4  D_4  E_4  F_4  G_4  A_4  B_4  C_5

261.63 Hz 293.66 Hz 329.63 Hz 349.23 Hz 392.00 Hz 440.00 Hz 493.88 Hz 523.25 Hz
**Consonance**

- Frequencies in consonance are neither similar enough to cause beats nor within the same critical band.

- Many of the overtones of these two frequencies coincide and most of the ones that don’t will neither cause beats nor be within the same critical band.

\[
\begin{array}{cccccc}
0 & f_1 & 2f_1 & 3f_1 & 4f_1 & 5f_1 & 6f_1 \\
0 & f_2 & 2f_2 & 3f_2
\end{array}
\]

**Ex 1:** \[f_2/f_1 = 2\]

Frequencies in consonance sound nearly the same.

<table>
<thead>
<tr>
<th>Frequency 1 (Hz)</th>
<th>Frequency 2 (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f_1 = 100)</td>
<td>(f_2 = 200)</td>
</tr>
<tr>
<td>(2f_1 = 200)</td>
<td>(2f_2 = 400)</td>
</tr>
<tr>
<td>(3f_1 = 300)</td>
<td>(3f_2 = 600)</td>
</tr>
<tr>
<td>(4f_1 = 400)</td>
<td></td>
</tr>
<tr>
<td>(5f_1 = 500)</td>
<td></td>
</tr>
<tr>
<td>(6f_1 = 600)</td>
<td></td>
</tr>
</tbody>
</table>
Consonance

Ex 2: \( f_2 / f_1 = 3/2 \)

Match of harmonics not quite as good, but the harmonics of \( f_2 \) that don’t match those of \( f_1 \) are still different enough from the harmonics of \( f_1 \) that no beats are heard and they don’t fall within the same critical band.

<table>
<thead>
<tr>
<th>Frequency 1 (Hz)</th>
<th>Frequency 2 (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_1 = 100 )</td>
<td>( f_2 = 150 )</td>
</tr>
<tr>
<td>( 2f_1 = 200 )</td>
<td>( 2f_2 = 300 )</td>
</tr>
<tr>
<td>( 3f_1 = 300 )</td>
<td>( 3f_2 = 450 )</td>
</tr>
<tr>
<td>( 4f_1 = 400 )</td>
<td>( 4f_2 = 600 )</td>
</tr>
<tr>
<td>( 5f_1 = 500 )</td>
<td></td>
</tr>
<tr>
<td>( 6f_1 = 600 )</td>
<td></td>
</tr>
</tbody>
</table>
Pythagorean scale

Ancient Greeks - Monochord most pleasant sounds with \( \frac{f_2}{f_1} = 2 \) and \( \frac{f_2}{f_1} = \frac{3}{2} \) \( \Rightarrow \) \( \frac{L_1}{L_2} = 2 \) and \( \frac{L_1}{L_2} = \frac{3}{2} \)

Building a scale (Pythagoras) - To get more “pleasant” tones multiply, or divide, strings by 3/2.

Problem: new string length might be shorter than the shortest string or longer than the longest string.

Solution: cut in half or double in length (even repeatedly) because strings that differ by a ratio of 2:1 sound virtually the same.
Building a Pythagorean scale

• Assume shortest string length = 1 \textbf{(whatever units)}. Longest one length = 2.

Let us start: \[ 1 \times \frac{3}{2} = \frac{3}{2} \] and \[ \frac{2}{3/2} = 2 \times \frac{2}{3} = \frac{4}{3} \]

\[ 1 \quad \frac{4}{3} \quad \frac{3}{2} \quad 2 \]

133 Hz 150 Hz

E.g. 100 Hz 200 Hz

This four-note scale is thought to have been used to tune ancient lyre
Let try more (using intermediate frequencies 4/3 and 3/2):

\[
\frac{4}{3} = \frac{4}{3} \times \frac{2}{3} = \frac{8}{9}
\]

and

\[
\frac{3}{2} \times \frac{3}{2} = \frac{9}{4}
\]

But 8/9 < 1 and 9/4 > 2. **Solution:** divide or multiply by 2, as they will sound nearly the same.

\[
\frac{8}{9} \times 2 = \frac{16}{9}
\]

and

\[
\frac{9}{4} / 2 = \frac{9}{8}
\]

**Pentatonic scale:** popular in many eastern cultures.
Building a Pythagorean scale - continued

Western Music has 7 notes → Let us continue:

\[
\frac{16}{9} = \frac{16}{9} \times \frac{2}{3} = \frac{32}{27}
\]

and

\[
\frac{9}{8} \times \frac{3}{2} = \frac{27}{16}
\]

One version of the **Pythagorean scale**.

• Many frequency ratios of small integers → high levels of consonance.

• Mostly large intervals, but also two small intervals (between the second and third note and between the sixth and seventh note).
• Across a large interval, the frequency must be multiplied by \( \frac{9}{8} \) (e.g., \( \frac{32}{27} \times \frac{9}{8} = \frac{4}{3} \) and \( \frac{4}{3} \times \frac{9}{8} = \frac{3}{2} \)).

• Across the smaller intervals, the frequency must be multiplied by \( \frac{256}{243} \). In fact, \( \frac{9}{8} \times \frac{256}{243} = \frac{32}{27} \), \( \frac{27}{16} \times \frac{256}{243} = \frac{16}{9} \).

• \( \frac{9}{8} = 1.125 = \text{change of } \approx 12\% \text{ (whole tone)} - W \) ("full step")
• \( \frac{256}{243} = 1.053 = \text{change of } \approx 5\% \text{ (semitone)} - s \) ("half step")

• Going up in frequency: \( WsWWWsW \)
Pythagorean scale

• Instead of $W s W W W s W$ let us start with previous $W$:

$W W s W W W s$

$c_1 \text{ D E F G A B C}_2$

Do Re Me Fa So La Ti Do ← (solfège)

• Nonmusicians do not notice the smaller increase in pitch when going from “Me” to “Fa” and from “Ti” to “Do.”

• 7 different notes in the Pythagorean scale (8 including last note, which is one diapason higher than the first note, and thus essentially the same sound as the first).

• The eighth note has a ratio of 2:1 with the first note, the fifth note has a ratio of 3:2 with the first note, and the fourth note has ratio of 4:3 with the first note.

• Origin of the musical terms the octave, perfect fifth, and perfect fourth.
Pythagorean scale

• “G” sounds good when played with either the upper or the lower “C.” It is a fifth above the lower C and a fourth below the upper C.

<table>
<thead>
<tr>
<th>C_i</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A</th>
<th>B</th>
<th>C_f</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9/8</td>
<td>81/64</td>
<td>4/3</td>
<td>3/2</td>
<td>27/16</td>
<td>243/128</td>
<td>2</td>
</tr>
</tbody>
</table>

• Multiplying the frequency of a particular “C” by one of the fractions in the table above gives the frequency of the note above that fraction.

• Table on right shows the full list of frequency intervals between adjacent tones.

• Exercise: Assuming C_5 is defined as 523 Hz, determine the other frequencies of the Pythagorean scale.
Just Scale  (origin: Ptolemy-Greece)

• Besides 2:1, 3:2 and 4:3, Ptolemy also observed consonance in frequency ratio 5:4. Ratios 4:5:6 sound particularly good → C major scale.

• Note in C scale are grouped in triads with frequency ratios 4:5:6

• Start with $C_i = 1 \rightarrow C_f = 2$. To get the $C_i$:E:G frequency ratios 4:5:6 represent $C_1$ as 4/4 → $E = 5/4$ and $G = 6/4$, or 3/2.

<table>
<thead>
<tr>
<th>$C_i$</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>A</th>
<th>B</th>
<th>$C_f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>5</td>
<td>3</td>
<td>4</td>
<td></td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Just Scale Intervals

• \(D=9/4 > 2xC_i\) → divide it by 2 to get it back within the octave bound by \(C_i\) and \(C_f\).

Then \(D = (9/4)/2 = 9/8\)

<table>
<thead>
<tr>
<th>(C_i)</th>
<th>(D)</th>
<th>(E)</th>
<th>(F)</th>
<th>(G)</th>
<th>(A)</th>
<th>(B)</th>
<th>(C_f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9/4</td>
<td>5/4</td>
<td>3/2</td>
<td>15/8</td>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• Last triad (F, A, C_f): easier to start with \(C_f\) backwards. To get the next set of 4:5:6 frequency ratios → multiply \(C_f\) by 4/6, 5/6, and 6/6 → \(F = 2x(4/6) = 8/6 = 4/3\), \(A = 2x(5/6) = 10/6 = 5/3\), \(C_f = 2x(6/6) = 12/6 = 2\).

Just Scale Intervals for a C major scale. Multiplying the frequency of a particular “C” by one of the fractions in the table gives the frequency of the note above that fraction.
### Just Scale Intervals

- **Just Scale interval ratios.**

There are three possible intervals between notes:

- 9/8 (a **major whole tone** = 12.5% increase - same as Pythagorean whole tone)
- 10/9 (a **minor whole tone** = 11.1% increase)
- 16/15 (a **semitone** = 6.7% increase - slightly different than smallest Pythagorean)

#### Just Scale Intervals and common names →

- **Exercise:** C₄ is the frequency or note one octave below C₅ (523 Hz). Calculate the frequencies of the notes in the Just scale within this octave.

<table>
<thead>
<tr>
<th>Note change</th>
<th>Frequency ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>C → D</td>
<td>9/8</td>
</tr>
<tr>
<td>D → E</td>
<td>10/9</td>
</tr>
<tr>
<td>E → F</td>
<td>16/15</td>
</tr>
<tr>
<td>F → G</td>
<td>9/8</td>
</tr>
<tr>
<td>G → A</td>
<td>10/9</td>
</tr>
<tr>
<td>A → B</td>
<td>9/8</td>
</tr>
<tr>
<td>B → C</td>
<td>16/15</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Frequency Ratio</th>
<th>Interval</th>
<th>Interval name</th>
</tr>
</thead>
<tbody>
<tr>
<td>2/1</td>
<td>C → C</td>
<td>Octave</td>
</tr>
<tr>
<td>3/2</td>
<td>C → G</td>
<td>Perfect fifth</td>
</tr>
<tr>
<td>4/3</td>
<td>C → F</td>
<td>Perfect fourth</td>
</tr>
<tr>
<td>5/3</td>
<td>C → A</td>
<td>Major sixth</td>
</tr>
<tr>
<td>5/4</td>
<td>C → E</td>
<td>Major third</td>
</tr>
<tr>
<td>8/5</td>
<td>E → C</td>
<td>Minor sixth</td>
</tr>
<tr>
<td>6/5</td>
<td>A → C</td>
<td>Minor third</td>
</tr>
</tbody>
</table>