Musical Acoustics

Lecture 17

Interval, Scales, Tuning and Temperament - II

Problems with Pythagorean and Just Scales

- Songs are not transposable

1 – E.g., a song is written in the key of C (meaning that it starts with the note, C).

2 - change it so that it is now written in the key of F.

→ *it wouldn't sound right.*

3 - it wouldn't be as easy as transposing all the notes in the song up by three notes (C → F, F → B, etc.) because of the difference in intervals between various notes.

Example: take notes C and F and rewrite them in F

First note (C→F): 9/8 • 9/8 • 256/243 = 1.33.

→ If song is to remain the same, every note should be multiplied by 1.33

But (F→B): 9/8 • 9/8 • 9/8 = 1.42.

Note change	Frequency ratio
C → D	9/8
D → E	9/8
E → F	256/243
F → G	9/8
G → A	9/8
A → B	9/8
B → C	256/243

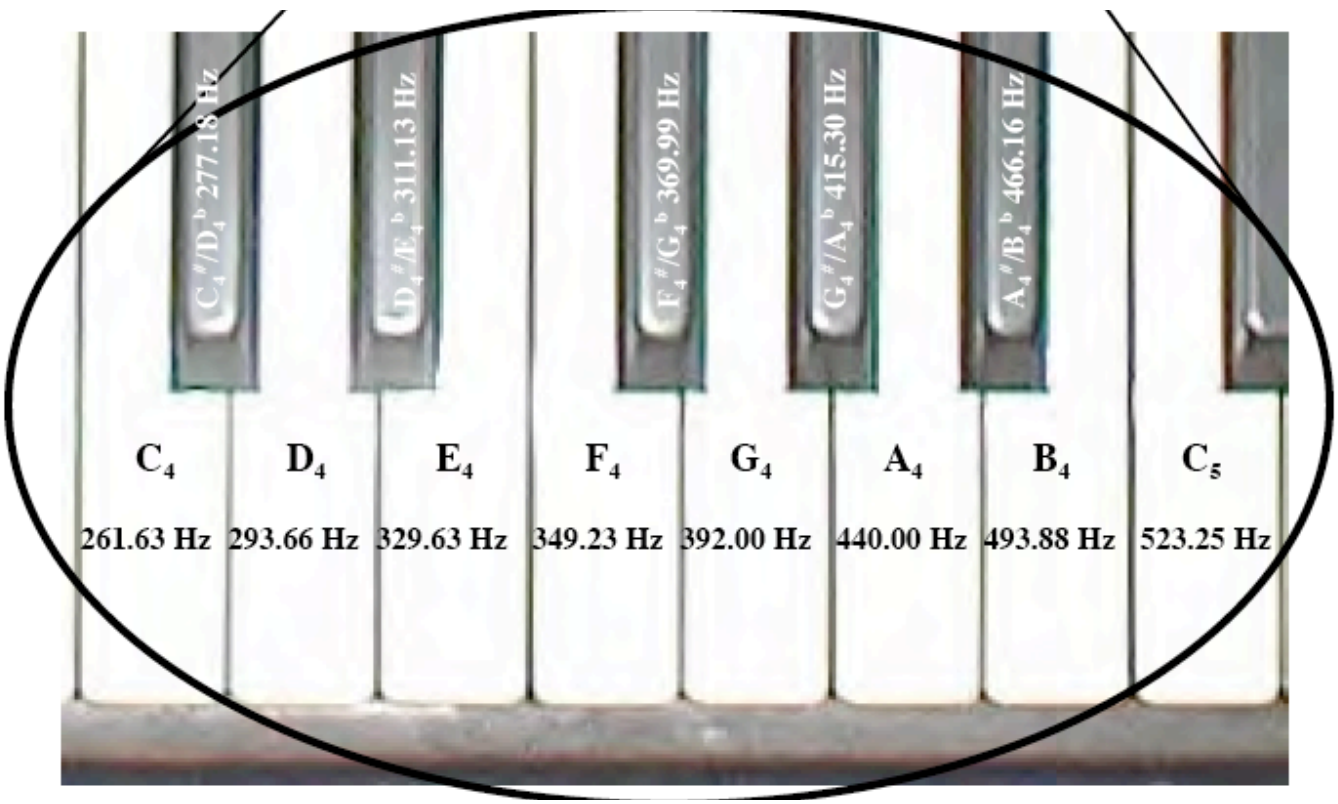
Just scale has same problem.

Sharps and flats

Seven white keys from C_4 to B_4 represent the **major diatonic scale**

Black keys are intermediate tones. E.g., black key in between D_4 and E_4 is higher frequency (sharper) than D_4 and lower frequency (flatter) than E4: "D4 sharp" ($D_4^{\#}$) or "E4 flat"(E_4^{b}).

Five sharps or flats + seven notes = full **chromatic scale**.



Sharps and flats in Pythagorean and Just Scales

- Attempt to rescue Pythagorean and Just Scales:

add extra notes (sharps and flats) between whole tones so that there were always choices for notes between the whole tones if needed for transposing.

→ Going from C to F or from F to B would be an increase of five semitones in both cases.

There is still the problem of the two different whole tone intervals in the Just scale.

Note change	Frequency ratio
C → D	9/8
D → E	10/9
E → F	16/15
F → G	9/8
G → A	10/9
A → B	9/8
B → C	16/15

In both scales, there is the problem that the semitone intervals are not exactly half the interval of the whole tones

Solution: Equal Temperament Scale

Include the seven notes of the previous scales, adding five sharps (for a total of 12 semitones), but placing them so that **the ratio of the frequencies of any two adjacent notes is the same.**

Multiplying the frequency of a note in the scale by a certain number *r* gives the frequency of the next note. Multiplying the frequency of this second note by the same number gives the frequency of the note following the second, and so on.

Going from a C to another C, means 12 multiplications:

$$r.r.r.r.r.r.r.r.r.r.r.r = 2$$

Equal Temperament Scale

$$r.r.r.r.r.r.r.r.r.r.r.r = 2$$

$$\Rightarrow \quad r^{12} = 2 \qquad\qquad r = \sqrt[12]{2} = 1.05946$$

Exercise: The note, D, is two semitones higher than C. If C_6 is 1046.5 Hz, what is D_7 on the Equal Temperament Scale?

Equal Temperament Scale

$$r.r.r.r.r.r.r.r.r.r.r.r = 2$$

$$\Rightarrow \quad r^{12} = 2 \qquad r = \sqrt[12]{2} = 1.05946$$

Exercise: The note, D, is two semitones higher than C. If C_6 is 1046.5 Hz, what is D_7 on the Equal Temperament Scale?

Solution: D_7 is one octave above D_6, so if D_6 can be found then its frequency just needs to be doubled. Since D_6 is two semitones higher than C_6, its frequency must be multiplied twice by the Equal temperament multiplier.

D_6 = 1046.5 Hz × $(1.05946)^2$ = 1174.7 Hz

→ $D_7 = 2D_6$ = 2(1174.7) Hz = 2349.3 Hz

Alternatively,

D_7 = 1046.5 Hz × $(1.05946)^{14}$ = 2349.3 Hz

Equal Temperament Scale

In Western music the Equal Temperament Scale is the most widely used. Its twelve semitones all differ in a ratio of $\sqrt[12]{2}$ from each adjacent semitone.

All notes of the major scale are separated by two semitones except for E and F, and B and C. These two pairs are separated by one semitone.

1	2	3	4	5	6	7	8	9	10	11	12
C	$C^\#$	**D**	$D^\#$	**E**	**F**	$F^\#$	**G**	$G^\#$	**A**	$A^\#$	**B**

The # indicates a "sharp."

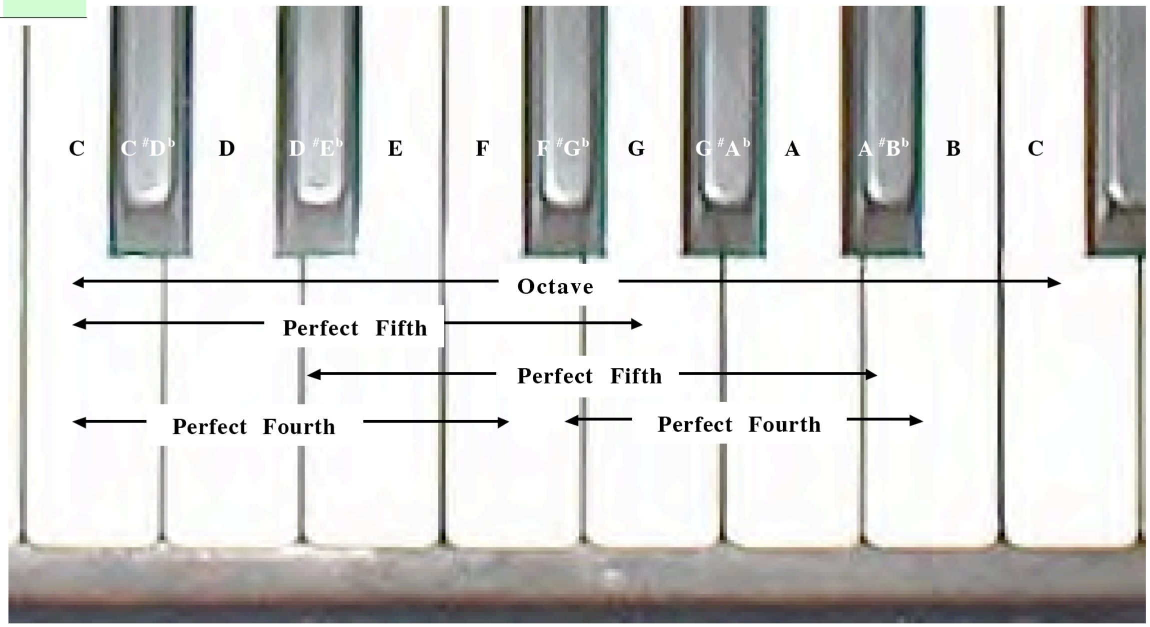
Equal Temperament Scale

$\left(\sqrt[12]{2}\right)^7 = 1.498$ is 0.1% different from 3/2 (the perfect fifth). Thus in the Equal Temperament scale, the seventh semitone note above any note will be close to a perfect fifth above it (sounds good).

C sounds good with the G above it, $D^{\#}$ also sounds just as good when played with the $A^{\#}$ above. The perfect fourth, occurs when two notes played together have a frequency ratio of 4/3. Note that $\left(\sqrt[12]{2}\right)^5 = 1.335$ (0.4% different from a perfect fourth).

1	2	3	4	5	6	7	8	9	10	11	12
C	$C^{\#}$	**D**	$D^{\#}$	**E**	**F**	$F^{\#}$	**G**	$G^{\#}$	**A**	$A^{\#}$	**B**

So the fifth semitone higher than any note will be higher by a virtual perfect fourth.



Cents

- One cent is 1/100 of an Equal Temperament semitone.
- 12 semitones per octave → one cent = 1/1200 of an octave.
- Any two frequencies that differ by one cent will have the same frequency ratio $R_{1\ cent}$.

$$R_{1\,\mathrm{cent}}^{1200} = 2 \quad \Rightarrow \quad R_{1\,\mathrm{cent}} = 2^{1/1200} = 1.000578$$

- Frequency ratio equivalent to two cents

$$R_{2\,\mathrm{cent}} = \left(2^{1/1200}\right)\left(2^{1/1200}\right) \quad \Rightarrow \quad R_{2\,\mathrm{cent}} = 2^{2/1200}$$

- Frequency ratio of I cents

$$R_{I\,\mathrm{cent}} = 2^{I/1200}$$

- Number of cents for a particular frequency ratio

$$\log R_I = \log\left(2^{I/1200}\right) \quad \Rightarrow \quad \log R_I = \frac{I}{1200}\log 2 \quad \Rightarrow \quad I = \frac{1200 \log R_I}{\log 2}$$

Cents

• A musician with a good ear can easily detect a mistuning of 5 cents and a 10 to 15 cent deviation from perfect intervals is enough to be unacceptable.

Just scale x Equal Temperament Scale

Interval	Frequency ratio	Frequency ratio (cents)	Equal Temp. scale (cents)
Octave	2 : 1	1200	1200
Fifth	3 : 2	702	700
Fourth	4 : 3	498	500
Major sixth	5 : 3	884	900
Major third	5 : 4	386	400
Minor sixth	8 : 5	814	800
Minor third	6 : 5	316	300

Note the large deviations from ideal for the thirds and sixths.

Example

- The perfect fifth is a frequency ratio of 1.5. How many cents is this and how does it compare with the fifths of the Just, Pythagorean, and Equal Tempered scale?

Example

- The perfect fifth is a frequency ratio of 1.5. How many cents is this and how does it compare with the fifths of the Just, Pythagorean, and Equal Tempered scale?

- Solution:

$$I = \frac{1200 \log R}{\log 2} = \frac{1200 \log(1.5)}{\log 2} = 702 \text{ cents}$$

Just scale:	identical
Pythagorean scale:	identical
Equal Tempered scale:	2 cents less

Comparison of scales

• Because of its equal intervals, the Equal Temperament scale makes transposing music very simple.

• **Example:** one can easily take a melody written in the key of C and rewrite it in the key of F by simply increasing every note's frequency by five semitones. This is a major advantage over the scales with unequal intervals.

• Although its octave is perfect and its fifth and fourth differ from the ideal by only 2 cents, the equal tempered sixths and thirds all differ from the ideal anywhere from 14 to 16 cents, clearly mistuned and noticeable by anyone with a good ear.

Pythagorean scale vs. cents

Note change	Frequency ratio
C → D	9/8
D → E	9/8
E → F	256/243
F → G	9/8
G → A	9/8
A → B	9/8
B → C	256/243

Pythagorean scale

$$I = \frac{1200 \log R}{\log 2}$$

Notes	Frequency interval (cents)
C_i	0
D	204
E	408
F	498
G	702
A	906
B	1110
C_f	1200

Pythagorean scale interval ratios expressed in cents

- Interval between C_i and G is 702 cents – a **perfect fifth**.
- Interval between D and A (906 cents – 204 cents = 702 cents) is also a **perfect fifth**.
- Additive nature of the cents unit makes it easy to judge the quality of various intervals.

Pythagorean scale vs. cents

An evaluation of the Pythagorean intervals. Note the abundance of perfect fifths and fourths, but also the very poorly tuned thirds.

Within the major scale, there are four perfect fifths and four perfect fourths (there are many more of both if the entire chromatic scale is used). However, the three major thirds differ by 22 cents (408 cents – 386 cents) from the perfect major third, noticeably sharp to most ears.

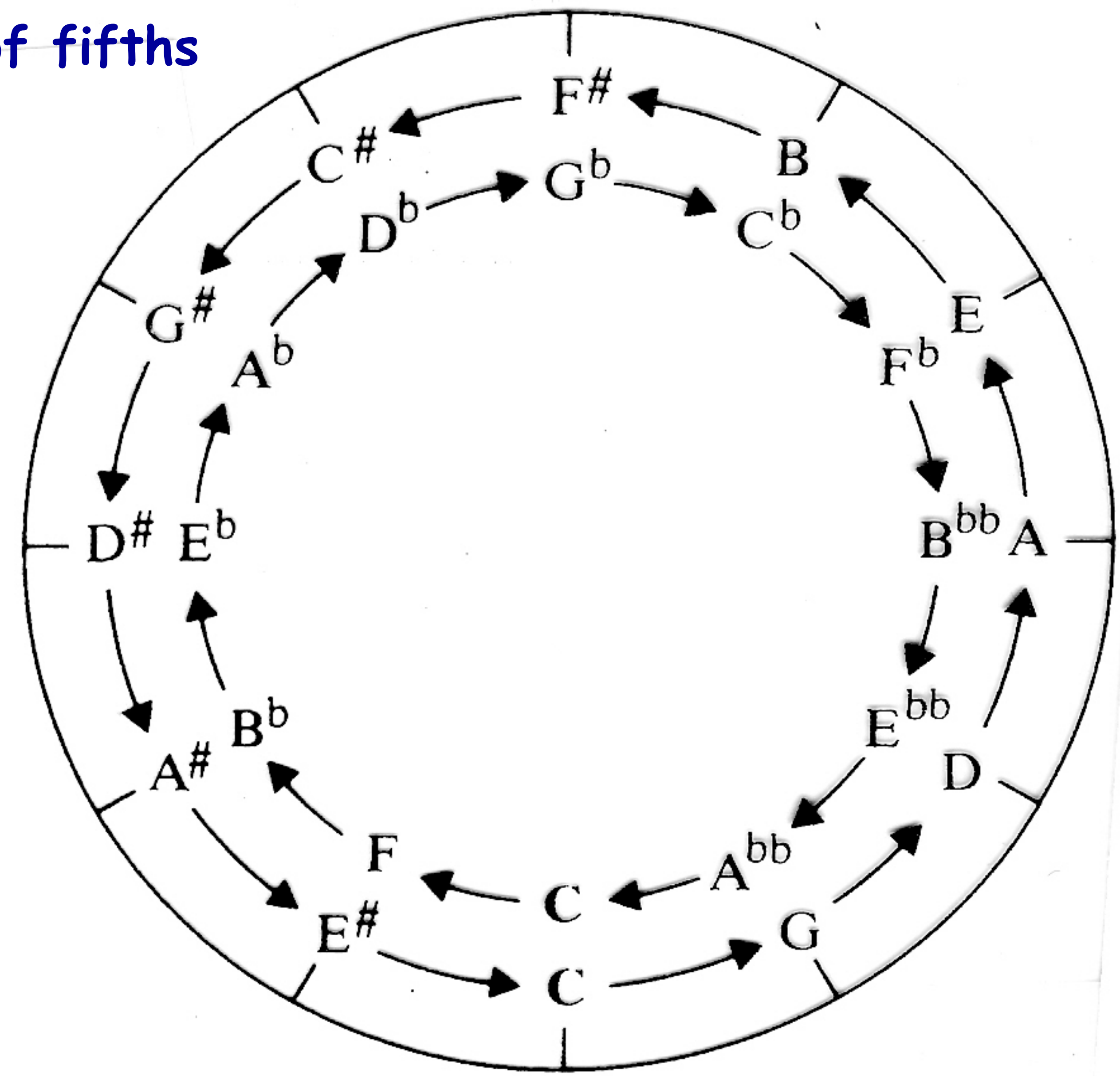
Interval	Interval name	Frequency ratio (cents)
$C_i \rightarrow C_f$	Octave	1200
$C_i \rightarrow G$	Fifth	702
$D \rightarrow A$	Fifth	702
$E \rightarrow B$	Fifth	702
$F \rightarrow C$	Fifth	702
$C_i \rightarrow F$	Fourth	498
$D \rightarrow G$	Fourth	498
$E \rightarrow A$	Fourth	498
$G \rightarrow C_f$	Fourth	498
$C_i \rightarrow E$	Major third	408
$F \rightarrow A$	Major third	408
$G \rightarrow B$	Major third	408

This is the reason that Pythagoras felt the major third was dissonant.

Pythagorean scale

Creates the greatest number of perfect fourths and fifths

Circle of fifths



All notes can be reached by going up (or down) 12 fifths or 12 fourths

$(3/2)^{12}$ = 129.7

but

2^7 = 128,

so

12 perfect fifths

is

$7\frac{1}{4}$ octaves

syntonic comma, δ

The minor third is a problem in the Pythagorean scale as well.

Going from E to G is an increase of 294 cents, but the perfect minor third, 6/5, is 316 cents.

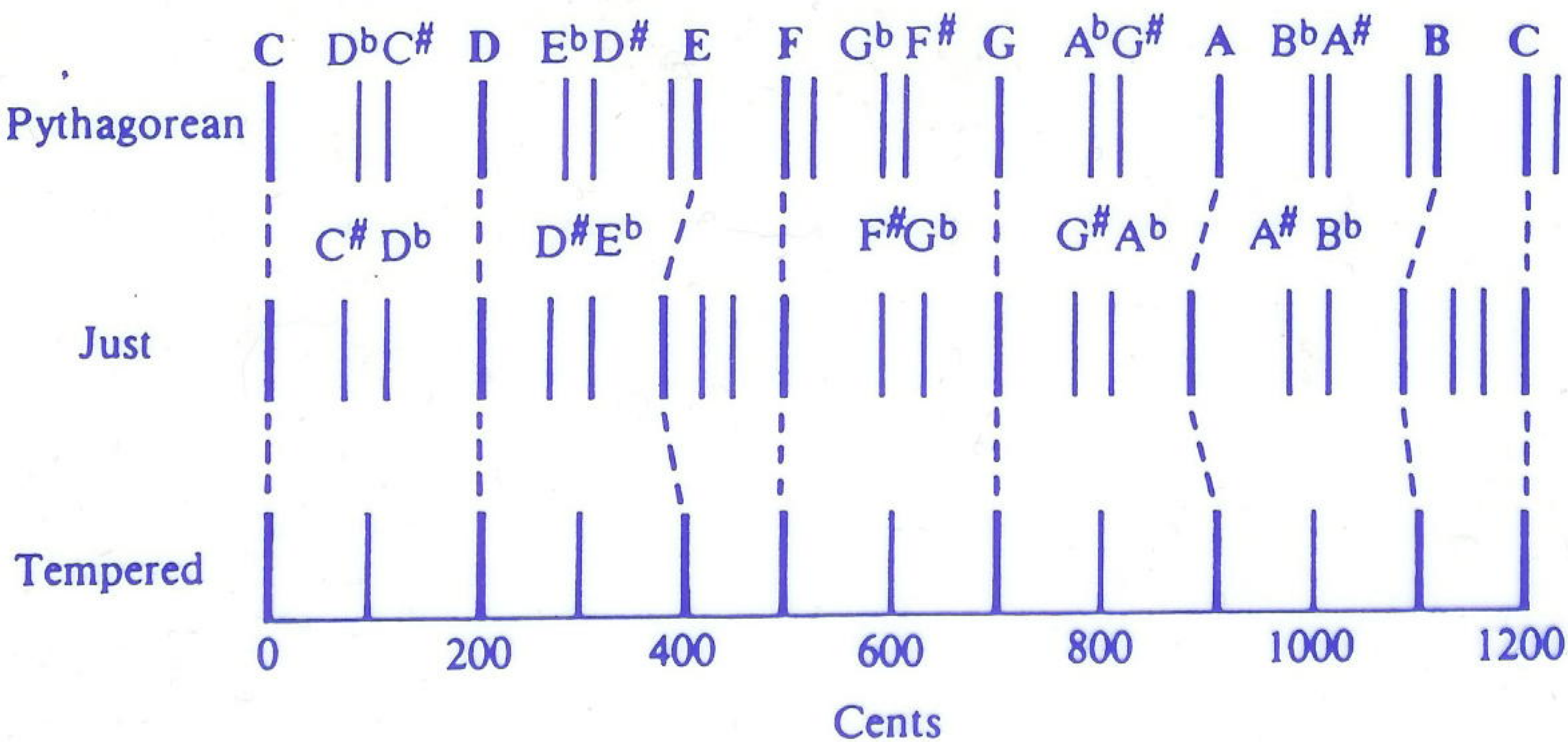
So the Pythagorean major third is 22 cents sharp and the minor third is flat by the same amount.

This 22-cent interval is actually 21.5 cents (due to rounding errors) and is known as the **syntonic comma, δ**.

Notes	Frequency interval (cents)
C_i	0
D	204
E	408
F	498
G	702
A	906
B	1110
C_f	1200

Interval	Frequency ratio	Frequency ratio (cents)	Equal Temp. scale (cents)
Octave	2 : 1	1200	1200
Fifth	3 : 2	702	700
Fourth	4 : 3	498	500
Major sixth	5 : 3	884	900
Major third	5 : 4	386	400
Minor sixth	8 : 5	814	800
Minor third	6 : 5	316	300

Comparison of Pythagorean, Just, and Equal Temperament scales on a scale of Cents



Musical intervals in various tunings

Interval	Tempered Ratio	Tempered Cents	Just Ratio	Just Cents	Pythagorean Ratio	Pythagorean Cents
Octave	2.000	1200	2/1 = 2.000	1200	2.000	1200
Fifth	1.498	700	3/2 = 1.500	702	1.500	702
Fourth	1.335	500	4/3 = 1.333	498	1.333	498
Major third	1.260	400	5/4 = 1.250	386	1.265	408
Minor third	1.189	300	6/5 = 1.200	316	1.184	294
Major sixth	1.682	900	5/3 = 1.667	884	1.687	906
Minor sixth	1.587	800	8/5 = 1.600	814	1.580	792

Meantone

Attempt: compromise the position of the E. Decreasing it a bit would help the consonance of both the major third (C to E) and the minor third (E to G).

Similar adjustments made to other notes in the scale are known as **meantone tuning**.

Notes	Frequency interval (cents)
C_i	0
D	204
E	408
F	498
G	702
A	906
B	1110
C_f	1200

There are different types of meantone tuning, but the most popular appears to be **quarter-comma meantone tuning**.

In this version, every note, except for C, is adjusted by either 1/4, 2/4, 3/4, 4/4, or 5/4 of the syntonic comma.

Interval	Frequency ratio	Frequency ratio (cents)	Equal Temp. scale (cents)
Octave	2 : 1	1200	1200
Fifth	3 : 2	702	700
Fourth	4 : 3	498	500
Major sixth	5 : 3	884	900
Major third	5 : 4	386	400
Minor sixth	8 : 5	814	800
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Quarter-comma meantone tuning

Pythagorean Note	Quarter-comma Meantone adjustment
C	none
D	$-\frac{2}{4}\delta$
E	$-\delta$
F	$+\frac{1}{4}\delta$
G	$-\frac{1}{4}\delta$
A	$-\frac{3}{4}\delta$
B	$-\frac{5}{4}\delta$
C	none

Summary of meantone tunnings:

Meantone tuning:

Decreasing C a bit to help the consonance of both the major third (C to E) and the minor third (E to G).

Similar adjustments made to other notes in the scale.

Quarter-comma meantone tuning.

In this version, every note, except for C, is adjusted by either 1/4, 2/4, 3/4, 4/4, or 5/4 of the syntonic comma.