Multilateral basic hypergeometric summation identities and hyperoctahedral group symmetries

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Abstract

We give new proofs for certain bilateral basic hypergeometric summation formulas using the symmetries of the corresponding series. In particular, we present a proof for Bailey's ${}_3\psi_3$ summation formula as an application. We also prove a multiple series analogue of this identity considering hyperoctahedral group symmetries of higher ranks.

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1. Introduction

Let $(a;q)_\alpha$ denote the q–Pochhammer symbol which is formally defined by

$$(a)_\alpha = (a;q)_\alpha := \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{1}$$

where the parameters $a, q, \alpha \in \mathbb{C}$, and $(a;q)_\infty$ denotes the infinite product $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$. Note here that when $\alpha = k$ is a positive integer, then the q–Pochhammer symbol reduces to $(a)_k = \prod_{i=1}^{k}(1-aq^{i-1})$. We often use the shorthand notation $(a_1, a_2, \ldots, a_r)_\alpha$ for the product $\prod_{i=1}^{r}(a_i)_\alpha$.

The series $\sum_{k=0}^{\infty} c_k$, where the ratio c_{k+1}/c_k is a rational function of q^k, is called a basic hypergeometric series [1]. Using the q–Pochhammer symbol (1), the general basic hypergeometric series with r numerator parameters and s denominator parameters is defined by

$${}_r\varphi_s\left[\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, b_2, \ldots, b_s\end{matrix}; x, q\right] := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_k}{(q, b_1, b_2, \ldots, b_s)_k}\, x^k \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \tag{2}$$

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where we assume that none of the denominator factors vanish.

Note that if one of the numerator parameters is of the form q^{-n}, for some non–negative integer n and $q \neq 0$, the series terminates from above since $(q^{-n})_k = 0$ when $k > n$. The denominator factor $(q)_k$ terminates the series from below, that is the factor $1/(q)_n$ causes the sum to vanish when $n < 0$.

In general, when dealing with non-terminating series it is assumed for convergence that $|q| < 1$. In that case, the series ${}_{r+1}\varphi_r$ converges absolutely for $|x| < 1$.

When $r = s + 1$, the basic hypergeometric series (2) is called well–poised if the parameters satisfy the relation

$$qa_1 = a_2 b_1 = a_3 b_2 = \ldots = a_{s+1} b_s,$$

and very well–poised if, in addition, $a_2 = q\sqrt{a_1}$ and $a_3 = -q\sqrt{a_1}$. An ${}_{r+1}\varphi_r$ series is called k-balanced if $b_1 \ldots b_r = q^k a_1 \ldots a_{r+1}$, and $x = q$.

There are numerous classical one-dimensional results, summation and transformation formulas for basic hypergeometric series. One of the most general summation formulas, for example, is the (q–Dougall or) Jackson sum

$$
{}_8\varphi_7 \left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq^{n+1} \end{matrix} ; q, q \right]
= \frac{(aq, aq/bc, aq/bd, aq/cd)_n}{(aq/b, aq/c, aq/d, aq/bcd)_n} \tag{3}
$$

where $qa^2 = bcdeq^{-n}$. An important general transformation formula is Bailey's ${}_{10}\varphi_9$ transformation

$$
\begin{aligned}
&{}_{10}\varphi_9 \left[\begin{matrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1} \end{matrix} ; q, q \right] \\
&= \frac{(aq, aq/ef, \lambda q/e, \lambda q/f)_\infty}{(aq/e, aq/f, \lambda q, \lambda q/ef)_\infty} \\
&\cdot {}_{10}\varphi_9 \left[\begin{matrix} \lambda, q\lambda^{1/2}, -q\lambda^{1/2}, \lambda b/a, \lambda c/a, \lambda d/a, e, f, \lambda a q^{n+1}/ef, q^{-n} \\ \lambda^{1/2}, -\lambda^{1/2}, aq/b, aq/c, aq/d, \lambda q/e, \lambda q/f, efq^{-n}/a, \lambda q^{n+1} \end{matrix} ; q, \frac{aq}{ef} \right]
\end{aligned} \tag{4}
$$

where $\lambda = qa^2/bcd$.

The basic hypergeometric series (2) is Heine's generalization of the hypergeometric series

$$
{}_rF_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; x \right] = \sum_{n=0}^{\infty} \frac{\{a_1\}_n \{a_2\}_n \ldots \{a_r\}_n}{n! \, \{b_1\}_n \{b_2\}_n \ldots \{b_s\}_n} x^n \tag{5}
$$

where $\{a\}_n$ denotes the shifted factorial (or Pochhammer symbol) defined by

$$\{a\}_0 := 1, \quad \{a\}_n := a(a+1)\cdots(a+n-1) \text{ for } n \in \mathbb{Z}_>. \tag{6}$$

The basic hypergeometric series (2) reduces to (5) if we replace parameters a_i and b_i by q^{a_i} and q^{b_i} in (2) respectively, and let $q \to 1$.

The basic hypergeometric series are further generalized in the literature in several directions. Bilateral basic hypergeometric series is a generalization where the index of summation is no longer restricted to non–negative integers, but it runs over all integers. The most general result of this type is Bailey's ${}_6\psi_6$ summation formula which can be written as

$$
\begin{aligned}
&{}_6\psi_6\left[\begin{matrix} qa^{1/2}, -qa^{1/2}, b, c, d, e \\ a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e \end{matrix}; q, \frac{qa^2}{bcde}\right] \\
&\qquad = \frac{(aq, aq/bc, aq/bd, aq/be, aq/cd, aq/ce, aq/de, q, q/a)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/b, q/c, q/d, q/e, qa^2/bcde)_\infty} \qquad (7)
\end{aligned}
$$

provided that $\left|qa^2/bcde\right| < 1$, where

$$
\begin{aligned}
&{}_r\psi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; x, q\right] \\
&\qquad = \sum_{n=-\infty}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_r)_n}{(b_1)_n (b_2)_n \ldots (b_s)_n} (-1)^{(s-r)n} q^{(s-r)\binom{n}{2}} x^n \qquad (8)
\end{aligned}
$$

There are other important summation formulas such as Ramanujan's ${}_1\psi_1$ sum, and useful transformation formulas for bilateral series as well. The former identity, for example, may be written as

$$
{}_1\psi_1(a, b; x, q) = \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} x^n = \frac{(q, b/a, ax, q/ax)_\infty}{(b, q/a, x, b/ax)_\infty} \qquad (9)
$$

Appell and Lauricella series [12] are other generalizations of the basic hypergeometric series, where the number of variables in the argument is extended to several variables and thus multisums are considered.

Macdonald [7] generalized the basic hypergeometric series to a multiple basic hypergeometric series of symmetric function argument where the argument is replaced by a Schur function, which is the ratio of two determinants, and the index of summation runs over partitions. Several summation and transformation results have been obtained at this generality as well. But all such results generalized lower level identities which satisfy only the balancedness condition.

The study of elliptic (modular) analogue of basic hypergeometric series was started by Frenkel and Turaev [6] who, defining an elliptic analogue of the q–Pochhammer symbol, proved an elliptic analogue of the ${}_{10}\varphi_9$ transformation given above. Several other authors contributed by proving various one dimensional and a few multisum identities including [13] and [11].

The elliptic generalization of the q–Pochhammer symbol is given by means of the normalized elliptic function

$$
\theta(x) = \theta(x; p) := (x; p)_\infty (p/x; p)_\infty \qquad (10)
$$

where $|p| < 1$. The elliptic q–Pochhammer symbol is then given by

$$
(a; q, p)_n = \prod_{k=0}^{n-1} \theta(aq^k) \qquad (11)
$$

for $n > 0$. The definition is extended to negative n by the relation $(a;q,p)_n = 1/(aq^n;q,p)_{-n}$ analogous to the standard q–Pochhammer symbol. When $n=0$, we have $(a;q,p)_n = 1$. Note also that when $p=0$ this reduces to standard definition of the q–Pochhammer symbol.

The definition of a balanced, very–well–poised elliptic basic hypergeometric series now may be written [13] as

$$ {}_{r+1}\omega_r(a_1;a_4,\ldots,a_{r+1};q,p) = \sum_{k=0}^{\infty} \frac{\theta(a_1q^{2k})}{\theta(a_1)} \frac{(a_1,a_4,\ldots,a_{r+1};q,p)_k q^k}{(q,a_1q/a_4,\ldots,a_1q/a_{r+1};q,p)_k} \tag{12} $$

where $(a_4\ldots a_{r+1})^2 = a_1^{r-3}q^{r-5}$. By defining the partition generalization of the elliptic q–Pochhammer symbol in the form

$$ (a)_\lambda = (a;q,p,t)_\lambda := \prod_{k=0}^{n-1} (at^{1-i};q,p)_{\lambda_i} \tag{13} $$

the definition of elliptic basic hypergeometric series is generalized to various root systems of rank n. The following shorthand notation will also be used.

$$ (a_1,\ldots,a_k)_\lambda = (a_1,\ldots,a_k;q,p,t)_\lambda := (a_1)_\lambda\ldots(a_k)_\lambda. \tag{14} $$

In [5] we proved a multiple elliptic analogue of the classical Jackson sum and other important results including a multiple analogue of Bailey's ${}_{10}\varphi_9$ transformation formula. The multiple elliptic Jackson sum may be written in the form

$$ \begin{aligned} & W_\lambda(z;q,p,t,at^{-2n},bt^{-n}) \\ &= \frac{(s)_\lambda(as^{-1}t^{-n-1})_\lambda}{(qbs^{-1}t^{-1})_\lambda(qbt^n s/a)_\lambda} \cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}(qbt^{-i-j+1})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}(qbt^{-i-j})_{\lambda_i+\lambda_j}} \right\} \\ &\cdot \sum_{\mu\subseteq\lambda} \frac{(bs^{-1}t^{-n})_\mu(qbt^n/a)_\mu}{(qt^{n-1})_\mu(as^{-1}t^{-n-1})_\mu} \cdot \prod_{i=1}^{n} \left\{ \frac{(1-bs^{-1}t^{1-2i}q^{2\mu_i})}{(1-bs^{-1}t^{1-2i})}(qt^{2i-2})^{\mu_i} \right\} \\ &\cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i})_{\mu_i-\mu_j}(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(t^{j-i+1})_{\mu_i-\mu_j}} \frac{(bs^{-1}qt^{-i-j})_{\mu_i+\mu_j}(bs^{-1}t^{-i-j+2})_{\mu_i+\mu_j}}{(bs^{-1}t^{-i-j+1})_{\mu_i+\mu_j}(qbs^{-1}t^{-i-j+1})_{\mu_i+\mu_j}} \right\} \\ &\cdot W_\mu(q^\lambda t^{\delta(n)};q,t,bt^{1-2n},bs^{-1}t^{-n}) \cdot W_\mu(zs;q,t,as^{-2}t^{-2n},bs^{-1}t^{-n}) \end{aligned} \tag{15} $$

where $z\in\mathbb{C}^n$ and W_λ denotes the symmetric Macdonald function [5] that is defined as follows.

Let $\lambda=(\lambda_1,\ldots,\lambda_n)$ and $\mu=(\mu_1,\ldots,\mu_n)$ be partitions of at most n parts for a positive integer n such that the skew partition λ/μ is a horizontal strip;

i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \lambda_n \geq \mu_n \geq \lambda_{n+1} = \mu_{n+1} = 0$. Setting

$$
\begin{aligned}
&H_{\lambda/\mu}(q,p,t,b) \\
&\quad := \prod_{1\leq i<j\leq n} \left\{ \frac{(q^{\mu_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j} (q^{\lambda_i+\lambda_j}t^{3-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j} (q^{\lambda_i+\lambda_j+1}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}} \right. \\
&\quad \left. \cdot \frac{(q^{\lambda_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j}}{(q^{\lambda_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j}} \right\} \cdot \prod_{1\leq i<(j-1)\leq n} \frac{(q^{\mu_i+\lambda_j+1}t^{1-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i+\lambda_j}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}} \qquad (16)
\end{aligned}
$$

we define

$$
\begin{aligned}
W_{\lambda/\mu}(x;q,p,t,a,b) &:= H_{\lambda/\mu}(q,p,t,b) \cdot \frac{(x^{-1},ax)_\lambda (qbx/t, qb/(axt))_\mu}{(x^{-1},ax)_\mu (qbx, qb/(ax))_\lambda} \\
&\cdot \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{1-2i}q^{2\mu_i})}{\theta(bt^{1-2i})} \frac{(bt^{1-2i})_{\mu_i+\lambda_{i+1}}}{(bqt^{-2i})_{\mu_i+\lambda_{i+1}}} \cdot t^{i(\mu_i-\lambda_{i+1})} \right\} \qquad (17)
\end{aligned}
$$

where $q,p,t,x,a,b \in \mathbb{C}$. Note that $W_{\lambda/\mu}(x;q,p,t,a,b)$ vanishes unless λ/μ is a horizontal strip. The symmetric function $W_{\lambda/\mu}(y,z_1,\ldots,z_\ell;q,p,t,a,b)$ is extended to $\ell+1$ variables $y,z_1,\ldots,z_\ell \in \mathbb{C}$ through the following recursion formula

$$
\begin{aligned}
&W_{\lambda/\mu}(y,z_1,z_2,\ldots,z_\ell;q,p,t,a,b) \\
&\qquad = \sum_{\nu\prec\lambda} W_{\lambda/\nu}(yt^{-\ell};q,p,t,at^{2\ell},bt^\ell)\, W_{\nu/\mu}(z_1,\ldots,z_\ell;q,p,t,a,b). \qquad (18)
\end{aligned}
$$

These functions generalize Macdonald polynomials and interpolation Macdonald polynomials [7, 9] and are closely related to BC_n abelian functions [10].

2. Multilateral Basic Series Identities

I would like to present our bilateralization argument first in one dimensional case to help make the multiple multilateral analogues easier to read. The classical Jackson sum, that is ${}_8\phi_7$ summation formula, for example, may be written in the form

$$
\begin{aligned}
&\sum_{k=0}^{n} \frac{(1-bq^{2k})}{(1-b)} \frac{(b,q^{-n})_k}{(q,bq^{1+n})_k} \frac{(\sigma,\rho,\gamma,b^2q^{1+n}/\sigma\rho\gamma)_k}{(qb/\sigma,qb/\rho,qb/\gamma,\sigma\rho\gamma q^{-n}/b)_k} q^k \\
&\qquad = \frac{(qb,qb/\sigma\rho,qb/\sigma\gamma,qb/\rho\gamma)_n}{(qb/\sigma,qb/\rho,qb/\gamma,qb/\sigma\rho\gamma)_n} \qquad (19)
\end{aligned}
$$

By using the definition (1) of q–Pochhammer symbol and the identity

$$
(a)_k = \frac{(-a)^k q^{\binom{k}{2}}}{(q/a)_{-k}} \qquad (20)
$$

we may flip factors and write the summand in the left hand side as

$$\frac{q^{-z^2}}{(q^{-2z})_\infty (q^{2z})_\infty} \frac{(\sigma, \rho, \sigma q^{-2z}, \rho q^{-2z})_\infty}{(q^{1-2z}, q^{1+n}, q, q^{1+n+2z})_\infty}$$
$$\cdot (\gamma, q^{4z+1+n}/\sigma\rho\gamma, \gamma q^{-2z}, q^{1+n+2z}/\sigma\rho\gamma)_\infty$$
$$\cdot \frac{(q^{1-z-(z+k)}, q^{1+n+z-(z+k)}, q^{1-z+(z+k)}, q^{1+n+z+(z+k)})_\infty}{(\sigma q^{-z+(k+z)}, \rho q^{-z+(k+z)}, \sigma q^{-z-(k+z)}, \rho q^{-z-(k+z)})_\infty}$$
$$\cdot \frac{q^{(z+k)^2}(q^{-2(z+k)})_\infty (q^{2(z+k)})_\infty}{(\gamma q^{-z+(k+z)}, q^{3z+1+n+(z+k)}/\sigma\rho\gamma, \gamma q^{-z-(k+z)}, q^{3z+1+n-(k+z)}/\sigma\rho\gamma)_\infty} \qquad (21)$$

where we also set $b = q^{2z}$ for some $z \in \mathbb{C}$. It is clear that the summand is invariant under the maps $(z+k) \leftrightarrow w(z+k)$ for all w in the hyperoctahedral group of rank 1, namely $\mathbb{Z}_2$. It is clear that these maps generate full weight lattice $\mathbb{Z}$ for the root system C_1 if $z = m/2$ as illustrated in [2] for Rogers–Selberg identity. However, we will only consider the case when $m = \delta \in \{0, 1\}$, that is $b = q^\delta$.

Recall also that the Macdonald polynomial identity [8] for the root system C_1 of rank 1 may be written as

$$1 = \frac{1}{1-x^2} + \frac{1}{1-x^{-2}} \qquad (22)$$

By setting $x = q^{(z+k)}$ and multiplying the sum on the left hand side by the polynomial identity and simplifying, we get

$$\sum_{k=0}^{n} \frac{(1-q^{\delta+2k})}{(1-b)} \frac{(q^\delta, q^{-n})_k}{(q, q^{\delta+1+n})_k} \frac{(\sigma, \rho, \gamma, q^{2\delta+1+n}/\sigma\rho\gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, \sigma\rho\gamma q^{-\delta-n})_k} q^k$$
$$= \sum_{k=-n-\delta}^{n} f(\delta) \frac{(q^{-n})_k}{(q^{\delta+1+n})_k} \frac{(\sigma, \rho, \gamma, q^{2\delta+1+n}/\sigma\rho\gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, \sigma\rho\gamma q^{-\delta-n})_k} q^k$$
$$= \frac{(q^{1+\delta}, q^{1+\delta}/\sigma\rho, q^{1+\delta}/\sigma\gamma, q^{1+\delta}/\rho\gamma)_n}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, q^{1+\delta}/\sigma\rho\gamma)_n} \qquad (23)$$

where

$$f(\delta) = \begin{cases} 1 & \text{when } \delta = 0 \\ 1/(1-q^\delta) & \text{when } \delta = 1 \end{cases} \qquad (24)$$

Now we send $n \to \infty$ applying the dominated convergence theorem for infinite series [3] to get

$$\sum_{k=-\infty}^{\infty} f(\delta) \left(\frac{q^{(\delta+1)}}{\sigma\rho\gamma}\right)^k \frac{(\sigma, \rho, \gamma)_k}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma)_k}$$
$$= \frac{(q^{1+\delta}, q^{1+\delta}/\sigma\rho, q^{1+\delta}/\sigma\gamma, q^{1+\delta}/\rho\gamma)_\infty}{(q^{1+\delta}/\sigma, q^{1+\delta}/\rho, q^{1+\delta}/\gamma, q^{1+\delta}/\sigma\rho\gamma)_\infty} \qquad (25)$$

Here we also used the limit rule

$$\lim_{a\to 0} a^k (x/a)_k = (-1)^k x^k q^{\binom{k}{2}} \tag{26}$$

Now, setting $\delta = 0$ gives Bailey's ${}_3\psi_3$ bilateral summation formula. The $\delta = 1$ case appears to be a new bilateral sum.

We now give a multilateral analogue of Bailey's ${}_3\psi_3$ bilateral summation formula. Recall [5] that when $z = xt^\delta$ for some $x \in \mathbb{C}$ and $t^\delta = (t^{n-1}, t^{n-2}, \ldots, t, 1)$, the multiple Jackson sum (15) may be written as

$$\frac{(sx^{-1}, asx)_\lambda}{(qbx, qb/ax)_\lambda} = \sum_{\mu\subseteq\lambda} q^{|\mu|} t^{2n(\mu)} \frac{(s, as)_\lambda}{(qb, qb/a)_\lambda} \frac{(bt^{1-n}, qb/as)_\mu}{(qt^{n-1}, as)_\mu}$$
$$\cdot \prod_{i=1}^{n} \left\{ \frac{(1 - bt^{2-2i} q^{2\mu_i})}{(1 - bt^{2-2i})} \right\} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (bt^{2-i-j})_{\mu_i+\mu_j}} \right\}$$
$$\cdot W_\mu(q^\lambda t^{\delta(n)}; q, t, bst^{2-2n}, bt^{1-n}) \frac{(x^{-1}, ax)_\mu}{(qbx, qb/ax)_\mu} \tag{27}$$

It was also shown [4] that the summand which includes the W_μ function is invariant under the hyperoctahedral group action of permutations an sign changes for arbitrary partitions λ. More precisely, it was shown that under the specialization $t = q^k$ and $b = q^{2z_i+2k(i-1)}$ where $z_i \in \mathbb{C}$ and $k \ge 0$ is a non–negative integer, the summand is invariant under the action $(\mu_i + z_i) \leftrightarrow w(\mu_i + z_i)$ for all elements $w \in W$, the hyperoctahedral group or rank n. It was further verified that this action generates the full weight lattice $\mathbb{Z}^n$ only if $z_i = m/2 + k(n-i)$ for some non–negative integers $m, k \ge 0$.

The proof of the invariance follows from the duality formula and the flip identity [2] for W_λ functions. The symmetry under permutations are given by duality formula

$$W_\lambda \left(k^{-1} q^\nu t^\delta; q, t, k^2 a, kb\right) \cdot \frac{(qbt^{n-1})_\lambda (qb/a)_\lambda}{(k)_\lambda (kat^{n-1})_\lambda}$$
$$\cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i})_{\lambda_i-\lambda_j} (qa't^{2n-i-j-1})_{\lambda_i+\lambda_j}}{(t^{j-i+1})_{\lambda_i-\lambda_j} (qa't^{2n-i-j})_{\lambda_i+\lambda_j}} \right\}$$
$$= W_\nu \left(h^{-1} q^\lambda t^\delta; q, t, h^2 a', hb\right) \cdot \frac{(qbt^{n-1})_\nu (qb/a')_\nu}{(h)_\nu (ha't^{n-1})_\nu}$$
$$\cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i})_{\nu_i-\nu_j} (qat^{2n-i-j-1})_{\nu_i+\nu_j}}{(t^{j-i+1})_{\nu_i-\nu_j} (qat^{2n-i-j})_{\nu_i+\nu_j}} \right\} \tag{28}$$

where $k = a't^{n-1}/b$ and $h = at^{n-1}/b$. The invariance under sign changes follows from the flip identity

$$a^{|\lambda|} b^{-|\lambda|} q^{-|\lambda|} t^{-n(\lambda)+(n-1)|\lambda|} W_\lambda(x_1^{-1}, \ldots, x_n^{-1}; q^{-1}, p, t^{-1}, a^{-1}, b^{-1})$$
$$= a^{-|\lambda|} b^{|\lambda|} q^{|\lambda|} t^{n(\lambda)-(n-1)|\lambda|} W_\lambda(x_1, \ldots, x_n; q, p, t, a, b) \tag{29}$$

The invariance of other factors follows immediately form the definition of the q–Pochhammer symbol.

We will give the multiple ${}_3\psi_3$ summation using the specialization $m = \delta \in \{0, 1\}$ as in the classical one dimensional case, and for $k = 1$ or $t = q$. In other words, we let $t \to q$ and $b \to q^{\delta+2(n-1)}$ and write the identity above in the form

$$\begin{aligned}
&\frac{(sx^{-1}, asx)_\lambda}{(s, as)_\lambda} \frac{(q^{\delta+2n-1}, q^{\delta+2n-1}/a)_\lambda}{(q^{\delta+2n-1}x, q^{\delta+2n-1}/ax)_\lambda} \\
&\qquad = \sum_{\mu\in\mathbb{Z}^n} f(\delta)\, q^{|\mu|+2n(\mu)} \frac{(q^{\delta+2n-1}/as)_\mu}{(as)_\mu} \\
&\qquad \cdot \prod_{1\le i<j\le n} \left\{ \frac{(q^{j-i+1})_{\mu_i-\mu_j} (q^{\delta+2n+1-i-j})_{\mu_i+\mu_j}}{(q^{j-i})_{\mu_i-\mu_j} (q^{\delta+2n-i-j})_{\mu_i+\mu_j}} \right\} \\
&\qquad \cdot W_\mu(q^{\lambda+\delta(n)}; q, q, sq^\delta, q^{\delta+(n-1)}) \frac{(x^{-1}, ax)_\mu}{(q^{\delta+2n-1}x, q^{\delta+2n-1}/ax)_\mu}
\end{aligned} \tag{30}$$

where

$$f(\delta) := \frac{1}{2^n} \prod_{i=1}^{n-1} \frac{1}{(1+q^{n-i})}, \quad \text{if } \delta = 0 \tag{31}$$

and

$$f(\delta) := \frac{1}{2^n} \prod_{i=1}^{n} \frac{1}{(1-q^{1+2n-2i})}, \quad \text{if } \delta = 1 \tag{32}$$

Note also that although the series is written over $\mathbb{Z}^n$, it actually terminates from above by λ and from below by $(-\lambda_i - 2n - 2i + \delta)$.

The analogue of Weyl degree formula [4] for W_μ functions implies that

$$\begin{aligned}
&W_\mu(q^{N+\delta(n)}; q, q, sq^\delta, q^{\delta+(n-1)}) \\
&\quad = \frac{(q^{-N}, sq^{\delta+N+n-1})_\mu}{(q^{N+\delta+2n-1}, q^{-N+n}/s)_\mu} \prod_{1\le i<j\le n} \frac{(q^{j-i+1})_{\mu_i-\mu_j} (q^{\delta+2n-i-j+1})_{\mu_i+\mu_j}}{(q^{j-i})_{\mu_i-\mu_j} (q^{\delta+2n-i-j})_{\mu_i+\mu_j}}
\end{aligned} \tag{33}$$

Therefore, by setting $\lambda = N^n = (N, N, \ldots, N)$ and sending $N \to \infty$ we get

$$\begin{aligned}
&\frac{(sx^{-1}, asx)_{\infty^n}}{(s, as)_{\infty^n}} \frac{(q^{\delta+2n-1}, q^{\delta+2n-1}/a)_{\infty^n}}{(q^{\delta+2n-1}x, q^{\delta+2n-1}/ax)_{\infty^n}} \frac{1}{2^n f(\delta)} \\
&\quad = \sum_{\mu\in\mathbb{Z}^n} q^{(1-n)|\mu|+2n(\mu)} s^{|\mu|} \frac{(q^{\delta+2n-1}/as)_\mu}{(as)_\mu} \frac{(x^{-1}, ax)_\mu}{(q^{\delta+2n-1}x, q^{\delta+2n-1}/ax)_\mu} \\
&\quad \cdot \prod_{1\le i<j\le n} \left\{ \frac{(1-q^{j-i+\mu_i-\mu_j})}{(1-q^{j-i})^2} \frac{(1-q^{\delta+2n-i-j+\mu_i+\mu_j})}{(1-q^{\delta+2n-i-j})^2} \right\}
\end{aligned} \tag{34}$$

This is the multilateral analogue of Bailey's bilateral ${}_3\psi_3$ summation formula as desired.

3. Conclusion

The multilateralization technique applied here can be used to prove other multilateral series when the invariance property is satisfied. We will explore similar identities in other publications.

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