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AN ELLIPTIC BC_n BAILEY LEMMA, MULTIPLE ROGERS–RAMANUJAN IDENTITIES AND EULER'S PENTAGONAL NUMBER THEOREMS

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Abstract. An elliptic BC_n generalization of the classical two parameter Bailey Lemma is proved, and a basic one parameter BC_n Bailey Lemma is obtained as a limiting case. Several summation and transformation formulas associated with the root system BC_n are proved as applications, including a ${}_6\varphi_5$ summation formula, a generalized Watson transformation and an unspecialized Rogers–Selberg identity. The last identity is specialized to give an infinite family of multilateral Rogers–Selberg identities. Standard determinant evaluations are then used to compute B_n and D_n generalizations of the Rogers–Ramanujan identities in terms of determinants of theta functions. Starting with the BC_n ${}_6\varphi_5$ summation formula, a similar program is followed to prove an infinite family of D_n Euler's Pentagonal Number Theorems.

1. Introduction

The Rogers–Ramanujan identities and Euler's Pentagonal Number Theorem are decisively among the most celebrated classical q–series identities. These identities are usually written in terms of the q–Pochhammer symbol $(a;q)_\alpha$ for $q,\alpha \in \mathbb{C}$, which is defined formally as

$$(a)_\alpha = (a;q)_\alpha := \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \tag{1.1}$$

where $(a;q)_\infty := \prod_{i=0}^{\infty}(1-aq^i)$.

With this notation, the Rogers–Ramanujan identities can be written in the form

$$\sum_{m=0}^{\infty} \frac{q^{m(m+\delta)}}{(q;q)_m} = \frac{1}{(q^{1+\delta};q^5)_\infty (q^{4-\delta};q^5)_\infty} \tag{1.2}$$

where $\delta \in \{0,1\}$ and $|q|<1$. These identities have a very rich history. Many important figures in mathematics had contributed to the development of these identities starting with Rogers [34] who first proved them in 1894, and Ramanujan [24] whose involvement made Rogers' unnoticed work popular. Others contributed by simplifying existing proofs, suggesting new proofs of different nature, establishing their relations to other branches of mathematics and generalizing these identities [7], [8],

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[22], [30], [39], [40]. This paper proves multiple series analogues of the Rogers–Ramanujan identities associated to root systems B_n and D_n of rank n.

Section §3 reviews definitions of two remarkable families of BC_n type symmetric rational functions [13], namely Macdonald functions $W_{\lambda/\mu}(z;r,q,p,t;a,b)$ and Jackson coefficients $\omega_{\lambda/\mu}(z;r,q,p,t;a,b)$. Properties of these functions are then used to prove an elliptic BC_n generalization of the classical two parameter Bailey Lemma. In particular, an important elegant property called the cocycle identity for $\omega_{\lambda/\mu}$ played an important role in this program. It may be written in the form

$$
\begin{aligned}
(1.3)\quad & \omega_{\nu/\mu}((uv)^{-1};uv,q,p,t;a(uv)^2,buv) \\
&\quad = \sum_{\mu\subseteq\lambda\subseteq\nu} \omega_{\nu/\lambda}(v^{-1};v,q,p,t;a(vu)^2,bvu)\,\omega_{\lambda/\mu}(u^{-1};u,q,p,t;au^2,bu)
\end{aligned}
$$

where the summation index λ runs over partitions.

A basic one parameter BC_n Bailey Lemma is obtained as a limiting case of the two parameter BC_n Bailey Lemma in the same section. The one–parameter Bailey Lemma is iterated to generate several remarkable BC_n analogues of the classical basic hypergeometric series identities including a ${}_6\varphi_5$ summation formula and a generalized Watson transformation.

Section §4 gives a BC_n Rogers–Selberg identity as a limiting case of the BC_n Watson transformation. Specializing the parameters of this identity gives rise to an infinite family of multilateral Rogers–Selberg identities associated with the root systems B_n and D_n, using a very general multilateralization argument. Standard determinant evaluations are then used to compute the B_n and D_n generalizations of the Rogers–Ramanujan identities in terms of determinants of theta functions.

With the notation as above, D_n multiple Rogers–Ramanujan identities can be written as

$$
\begin{aligned}
(1.4)\quad & \sum_{\lambda\in\mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(\delta+n-1)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \left\{(1-q^{\lambda_i-\lambda_j})\right\} \\
&= \frac{(-1)^{\binom{n}{2}} q^{\binom{n}{2}(4n+3\delta-2)/6}}{2\,\theta(q;q^5)^n\,\theta(q^2;q^5)^n} \cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+1-4i+j};q^5) \\
&\qquad\qquad + q^{-(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+3-4i-j};q^5) \Big)
\end{aligned}
$$

where $\theta(z;q) := (z;q)_\infty(q/z;q)_\infty$. The cases $\delta=0$ and $\delta=1$ give the first and the second D_n Rogers–Ramanujan identities, respectively. A single B_n multiple Rogers–Ramanujan identity can be written, similarly, in the form

$$
\begin{aligned}
(1.5)\quad & \sum_{\lambda\in\mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(1+n)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \left\{(1-q^{\lambda_i-\lambda_j})\right\} \\
&= \frac{(-1)^{\binom{n}{2}+n} q^{n(n+1)(4n-1)/12}}{\theta(q;q^5)^n\,\theta(q^2;q^5)^n} \cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1/2)(n-i+1)}\theta(q^{6+4n-4i+j};q^5) \\
&\qquad\qquad - q^{-(j-1/2)(n-i+1)}\theta(q^{7+4n-4i-j};q^5) \Big)
\end{aligned}
$$

In both cases, n is a positive integer and $|q| < 1$ as usual. Alternative versions of (1.4) and (1.5) in form of determinant transformation identities are given in (4.66) and (4.67), respectively.

The classical Euler's Pentagonal Number Theorem states that

$$(q)_\infty = \sum_{m=0}^{\infty} (-1)^m q^{m+3\binom{m}{2}} \tag{1.6}$$

This beautiful identity was first proved by Euler [19] in 1793. Many generalizations and combinatorial interpretations appeared in literature [4], [5], [9], [18], [37], etc. This paper gives a remarkable multiple series analogue of Euler's Pentagonal Number Theorem associated to the root system D_n of rank n. In fact, Section §4 proves an infinite family of Euler's Pentagonal Number Theorems using the BC_n ${}_6\varphi_5$ summation formula from Section §3, and methods developed in that section. These identities can be written in the form

$$\begin{aligned}(q;q)_\infty^n \prod_{1\le i<j\le n} \frac{(q^{k(j-i)};q)_\infty}{(q^{k(j-i+1)};q)_\infty} &= \sum_{\mu\in\mathbb{Z}^n} (-1)^{|\mu|} q^{-kn(\mu)+3n(\mu')+|\mu|(k(n-1)+1)} \\ &\cdot \prod_{1\le i<j\le n}^{k,1} \left\{ \frac{(q^{1+k(j-i-1)+\mu_i-\mu_j};q)_\infty}{(q^{k(1+2n-i-j)+\mu_i+\mu_j};q)_\infty} \frac{(q^{1+k(-1+2n-i-j)+\mu_i+\mu_j};q)_\infty}{(q^{k(j-i+1)+\mu_i-\mu_j};q)_\infty} \right\}\end{aligned} \tag{1.7}$$

where $n \in \mathbb{Z}_>$, $m, k \in \mathbb{Z}_\geq$, and $|q| < 1$ as usual. Similar to (4.66) and (4.67), the $k = 1$ instance of (1.7) is written as a determinant identity in terms of theta functions in (4.77).

Some results of this paper, such as (1.4), were conjectured in author's PhD work [13] which was conducted under the supervision of Dr. R. A. Gustafson.

Section 2 outlines an alternative proof of a known result. This proof may be seen as a one dimensional version of the more general proofs used in higher dimensional results in later sections. It is hoped that this section helps to make the rest of the paper more readable.

2. A Generalization of Rogers–Ramanujan identities

Garrett, Ismail and Stanton [21] used orthogonal polynomials to obtain, among other results, the following generalization of the Rogers–Ramanujan identities.

$$\begin{aligned}\sum_{\lambda_i=0}^{\infty} \frac{q^{\lambda_i(\lambda_i+\delta)}}{(q;q)_{\lambda_i}} &= \frac{(-1)^\delta q^{-\binom{\delta}{2}} E_{\delta-2}(q)}{(q,q^4;q^5)_\infty} - \frac{(-1)^\delta q^{-\binom{\delta}{2}} D_{\delta-2}(q)}{(q^2,q^3;q^5)_\infty} \\ &= (-1)^\delta q^{-\binom{\delta}{2}} \frac{(q^5;q^5)_\infty}{(q)_\infty} \left[\theta(q^2;q^5)E_{\delta-2}(q) - \theta(q;q^5)D_{\delta-2}(q)\right]\end{aligned} \tag{2.1}$$

where $\delta \in \mathbb{Z}$ (also see [10]), and the Schur polynomials $D_\delta(q)$ and $E_\delta(q)$ are defined by

$$D_\delta(q) = D_{\delta-1} + q^\delta D_{\delta-2}, \quad D_0 = 1, D_1 = 1+q \tag{2.2}$$

$$E_\delta(q) = E_{\delta-1} + q^\delta E_{\delta-2}, \quad E_0 = 1, E_1 = 1 \tag{2.3}$$

recursively. In two special cases $\delta \in \{1, 0\}$ the formula (2.1) simplifies to give the classical Rogers–Ramanujan identities (1.2).

There are numerous (analytic, combinatorial, probabilistic, algebraic) proofs [2] of the Rogers–Ramanujan identities (1.2). Watson's proof [43] of these identities, for example, depends on the identity

$$
{}_8\varphi_7\left[\begin{matrix} b, qb^{1/2}, -qb^{1/2}, \sigma_1, \rho_1, \sigma_2, \rho_2, q^{-N} \\ b^{1/2}, -b^{1/2}, bq/\sigma_1, bq/\rho_1, bq/\sigma_2, bq/\rho_2, bq^{N+1} \end{matrix}; q, \frac{b^2q^{2+N}}{\sigma_1\sigma_2\rho_1\rho_2}\right]
= \frac{(bq, bq/\sigma_2\rho_2)_N}{(bq/\sigma_2, bq/\rho_2)_N}\,{}_4\varphi_3\left[\begin{matrix} bq/\sigma_1\rho_1, \sigma_2, \rho_2, q^{-N} \\ bq/\sigma_1, bq/\rho_1, \sigma_2\rho_2 q^{-N}/b \end{matrix}; q, q\right] \tag{2.4}
$$

called Watson transformation. Watson showed that in the limiting case as N, $\sigma_1,\ \sigma_2, \rho_1, \rho_2 \to \infty$, the transformation (2.4) gives the remarkable Rogers–Selberg identity

$$
\sum_{m=0}^{\infty} \frac{b^m q^{m^2}}{(q)_m} = \frac{1}{(qb)_\infty} \sum_{m=0}^{\infty} (-1)^m b^{2m} q^{m(5m-1)/2} \frac{(1-bq^{2m})}{(1-b)} \frac{(b)_m}{(q)_m} \tag{2.5}
$$

In special cases $b = q^\delta$ for $\delta \in \{1, 0\}$, the series on the right hand side of (2.5) can be written as a bilateral sum giving

$$
\sum_{m=0}^{\infty} \frac{q^{m(m+\delta)}}{(q)_m} = \frac{1}{(q)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{5\binom{m}{2}+2(\delta+1)m} \tag{2.6}
$$

The product representation (1.2) can now be computed applying the Jacobi triple product identity

$$
\theta(z;q) := (z, q/z; q)_\infty = \frac{1}{(q;q)_\infty} \sum_{m=-\infty}^{\infty} (-1)^m q^{\binom{m}{2}} z^m, \quad |q| < 1 \tag{2.7}
$$

to the right hand side of (2.6) after rescaling parameters q by q^5 and z by $q^{2(\delta+1)}$ for $\delta \in \{1, 0\}$.

In this section an alternative elementary proof of the identity (2.1) will be given using the symmetries of the Rogers–Selberg identity (2.5). A generalization of this argument will then be used in the proof of our multiple Rogers–Ramanujan identities below.

By setting $b = q^{2z}$ for some $z \in \mathbb{C}$, flipping appropriate factors using the definition of the q–Pochhammer symbol (1.1) and simplifying, the one dimensional Rogers–Selberg identity (2.5) can be written in the form

$$
\sum_{m=0}^{\infty} q^{(z+m)^2} (q^{1-z+(z+m)})_\infty = \sum_{m=0}^{\infty} q^{(z+m)^2} \frac{(q^{1-z+(z+m)})_\infty (q^{1-z-(z+m)})_\infty}{(q^{1+2(z+m)})_\infty (q^{1-2(z+m)})_\infty} \tag{2.8}
$$

The summand on the well–poised side is obviously invariant under the action of the Weyl group for C_1, that is $W = \mathbb{Z}_2 = \{\imath, w\}$, generated by the maps $q^{z+m} \leftrightarrow q^{\pm(z+m)}$ for any $z \in \mathbb{C}$. This implies that

$$
\sum_{m \in L_+} f(m,z) = \sum_{m \in wL_+} f(m,z) \tag{2.9}
$$

where f is the summand in the right hand side of (2.8), $L_+ = \imath L_+$ is the lattice of all non–negative integers and $wL_+ = \{-m - 2z : m \in L_+\}$.

The right hand side of the classical Rogers–Selberg identity (2.5) can be written in an equivalent form by flipping different terms as

$$\frac{1}{(q)_\infty}\sum_{m=0}^{\infty} q^{4zm}(-1)^m q^{2m+5\binom{m}{2}}\frac{(q^{2(z+m)})_\infty}{(q^{1+2(z+m)})_\infty}\frac{(q^{1+m})_\infty}{(q^{z+(z+m)})_\infty} \tag{2.10}$$

Now multiply the series (2.10) by the right hand side of the C_1 version of Macdonald's [29] polynomial identity (written slightly differently as)

$$1=\frac{(q^{1+2z+2m})_\infty}{(q^{2z+2m})_\infty}+\frac{(q^{1-2z-2m})_\infty}{(q^{-2z-2m})_\infty} \tag{2.11}$$

Since terms of the sum (2.10) as well as both terms of (2.11) are invariant under the maps $q^{z+m}\leftrightarrow q^{\pm(z+m)}$, the series (2.10) can be written in the form

$$\frac{1}{(q)_\infty}\sum_{m\in L_z}(-1)^m q^{2(1+2z)m+5\binom{m}{2}}\frac{(q^{1+m})_\infty}{(q^{2z+m})_\infty} \tag{2.12}$$

where $L_z=\cup_{w\in W} wL_+$. It is clear that the series (2.12) is possibly over the full weight lattice $L=\mathbb{Z}$ for C_1 (i.e., $L_z=L$) only when $z=\delta/2$ for some integer δ. In that case, one still needs to study possible overlaps and gaps between the lattices L_+ and wL_+.

Let's denote by $\mathcal{O}$ the "overlap" set $L_+\cap wL_+$ and by $\mathcal{G}$ the "gap" $L\backslash L_z$. Clearly, the overlap $\mathcal{O}\neq\emptyset$ and $\mathcal{G}=\emptyset$ if δ is non–positive, and $\mathcal{O}=\emptyset$ and $\mathcal{G}\neq\emptyset$ otherwise.

Assume that $\delta>0$. The series (2.12) can be written over L, since the summand $g(m,z)$ in (2.12) vanishes on $\mathcal{G}=\{-1,\dots,-2z+1\}=\{-1,\dots,-\delta+1\}$ due to the numerator factor $(q^{1+m})_\infty$. Note that the denominator factor $(q^{2z+m})_\infty\neq 0$ on $\mathcal{G}$. In particular, the second Rogers–Ramanujan identity corresponding to $\delta=1$ follows from the case $z=1/2$.

For $\delta\le 0$, the additional "overlap condition" needs to be verified. Namely,

$$\sum_{m\in\mathcal{O}} f(m,z)=\sum_{m\in\mathcal{O}} g(m,z) \tag{2.13}$$

where $\mathcal{O}=\{0,\dots,-2z\}=\{0,\dots,-\delta\}$. In particular, for $z=0$ the overlap is $\mathcal{O}=\{0\}$ and it is plain that $f(0,0)=g(0,0)$. The first Rogers–Ramanujan identity then follows using a rewriting of the BC_1 version of Macdonald's [29] polynomial identity

$$\frac{(-q^{1+z+m})_\infty}{(-q^{z+m})_\infty}+\frac{(-q^{1-z-m})_\infty}{(-q^{-z-m})_\infty}=1 \tag{2.14}$$

Multiplying the specialized Rogers–Selberg identity by Macdonald's polynomial identities (2.11) and (2.14) amounts to dropping certain factors corresponding to positive (or negative) roots in the summand.

It is already verified that the Rogers–Selberg identity (2.5) can be written as a bilateral sum when $b=q^\delta$ for all $\delta\in\mathbb{Z}_\geq$. Now defining a degree $\delta+1$ polynomial f_δ in x by

$$f_\delta(x):=(1-x^2q^\delta)(qx)_{\delta-1}\quad\text{and}\quad f_0(x):=(1+x), \tag{2.15}$$

and flipping appropriate terms, the right hand side of the Rogers–Selberg identity (2.5) may be written in the form

$$\frac{1}{2(q)_\delta}\sum_{m=-\infty}^{\infty}(-1)^m q^{2(1+\delta)m+5\binom{m}{2}} f_\delta(q^m) \tag{2.16}$$

Using the Jacobi triple product identity (2.7) one finally gets

$$\sum_{m=0}^{\infty} \frac{q^{m(m+\delta)}}{(q)_m} = \frac{(q^5;q^5)_\infty}{2(q;q)_\infty} \sum_{n=0}^{\delta+1} \frac{f_\delta^{(n)}(0)}{n!}\, \theta(q^{2(\delta+1)+n};q^5) \tag{2.17}$$

The coefficients $f_\delta^{(n)}(0)/n!$ can be easily computed via the terminating q-binomial theorem

$$(x;q)_\delta = \sum_{m=0}^{\delta} \begin{bmatrix} \delta \\ m \end{bmatrix}_q (-1)^m q^{\binom{m}{2}} x^m \tag{2.18}$$

in terms of q-binomial coefficients defined by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q)_n}{(q)_{n-m}(q)_m} \tag{2.19}$$

when $0 \le m \le n$, and 0 otherwise. The coefficients of f_δ yields a well–known alternative representation of Schur polynomials (2.2)

$$E_{\delta-2}(q) = \sum_k (-1)^k q^{k(5k-3)/2} \begin{bmatrix} \delta-1 \\ \lfloor \frac{\delta+1-5k}{2} \rfloor \end{bmatrix}_q \tag{2.20}$$

and

$$D_{\delta-2}(q) = \sum_k (-1)^k q^{k(5k+1)/2} \begin{bmatrix} \delta-1 \\ \lfloor \frac{\delta-1-5k}{2} \rfloor \end{bmatrix}_q . \tag{2.21}$$

for $\delta \ge 2$.

A slight generalization of this argument shows that (2.1) also holds when $\delta < 0$, and completes the proof of (2.1).

3. An Elliptic BC_n Bailey Lemma

In this section, an elliptic BC_n Bailey Lemma will be proved. This result will then be used in the next section to obtain multiple Rogers–Ramanujan identities associated to root systems.

First, definitions are given for infinite dimensional matrices $M(a,b)$ and $S(b)$ indexed by partitions with respect to partial inclusion ordering $\subseteq$ defined by

$$\mu \subseteq \lambda \Leftrightarrow \mu_i \le \lambda_i, \quad \forall i \ge 1. \tag{3.1}$$

As in [14], the $\mathbb{Z}$–space V denotes the space of infinite lower–triangular matrices whose entries are rational functions in complex parameters ρ_i, σ_i for $i \in \mathbb{Z}_\ge$ over the field $\mathbb{F} = \mathbb{C}(q,p,t,r,a,b)$. The condition that $u \in V$ is lower triangular with respect to the partial inclusion ordering (3.1) can be stated in the form

$$u_{\lambda\mu} = 0, \quad \text{when } \mu \not\subseteq \lambda. \tag{3.2}$$

The multiplication operation in V is defined by the relation

$$(uv)_{\lambda\mu} := \sum_{\mu\subseteq\nu\subseteq\lambda} u_{\lambda\nu} v_{\nu\mu} \tag{3.3}$$

for $u, v \in V$.

The definitions of $M(a,b)$ and $S(b)$ involve the symmetric elliptic Macdonald functions $W_{\lambda/\mu}$ and Jackson coefficients $\omega_{\lambda/\mu}$ on BC_n defined and investigated in [13] and [14]. A brief review of the definitions and basic properties of these functions are in order.

Recall that an elliptic analogue of the basic factorial is given in terms of $\theta(x)$ function as follows [20]. For $x, p \in \mathbb{C}$ and $|p| < 1$, let

$$\theta(x) = \theta(x;p) := (x;p)_\infty (p/x;p)_\infty \tag{3.4}$$

and for $a \in \mathbb{C}$, and a positive integer m define

$$(a;q,p)_m := \prod_{k=0}^{m-1} \theta(aq^m) \tag{3.5}$$

The definition is extended to negative m by setting $(a;q,p)_m = 1/(aq^m;q,p)_{-m}$. Note also that when $p = 0$, $(a;q,p)_m$ reduces to standard (trigonometric) q–Pochhammer symbol.

For any partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $t \in \mathbb{C}$, one can also define [42]

$$(a)_\lambda = (a;q,p,t)_\lambda := \prod_{k=1}^{n} (at^{1-i};q,p)_{\lambda_i}. \tag{3.6}$$

Note that when $\lambda = (\lambda_1) = \lambda_1$ is a single part partition, then $(a;q,p,t)_\lambda = (a;q,p)_{\lambda_1} = (a)_{\lambda_1}$. The following notation will also be used.

$$(a_1, \ldots, a_k)_\lambda = (a_1, \ldots, a_k; q,p,t)_\lambda := (a_1)_\lambda \ldots (a_k)_\lambda. \tag{3.7}$$

Now let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions of at most n parts for a positive integer n such that the skew partition λ/μ is a horizontal strip; i.e. $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_n \geq \mu_n$. Then for $q,p,t,x,a,b \in \mathbb{C}$, define

$$\begin{aligned} H_{\lambda/\mu}(q,p,t,b) := \prod_{1\leq i<j\leq n} \Bigg\{ & \frac{(q^{\mu_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j} (q^{\lambda_i+\lambda_j}t^{3-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j} (q^{\lambda_i+\lambda_j+1}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}} \\ & \cdot \frac{(q^{\lambda_i-\mu_{j-1}+1}t^{j-i-1})_{\mu_{j-1}-\lambda_j}}{(q^{\lambda_i-\mu_{j-1}}t^{j-i})_{\mu_{j-1}-\lambda_j}} \Bigg\} \cdot \prod_{1\leq i<(j-1)\leq n} \frac{(q^{\mu_i+\lambda_j+1}t^{1-j-i}b)_{\mu_{j-1}-\lambda_j}}{(q^{\mu_i+\lambda_j}t^{2-j-i}b)_{\mu_{j-1}-\lambda_j}} \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} W_{\lambda/\mu}(x;q,p,t,a,b) := H_{\lambda/\mu}(q,p,t,b) \cdot & \frac{(x^{-1},ax)_\lambda (qbx/t, qb/(axt))_\mu}{(x^{-1},ax)_\mu (qbx, qb/(ax))_\lambda} \\ & \cdot \prod_{i=1}^{n} \left\{ \frac{\theta(bt^{1-2i}q^{2\mu_i})}{\theta(bt^{1-2i})} \frac{(bt^{1-2i})_{\mu_i+\lambda_{i+1}}}{(bqt^{-2i})_{\mu_i+\lambda_{i+1}}} \cdot t^{i(\mu_i-\lambda_{i+1})} \right\}. \end{aligned} \tag{3.9}$$

For arbitrary λ and μ the function $W_{\lambda/\mu}(y, z_1, \ldots, z_\ell; q,p,t,a,b)$ in $\ell+1$ variables $y, z_1, \ldots, z_\ell \in \mathbb{C}$ is defined by the following recursion formula

$$\begin{aligned} & W_{\lambda/\mu}(y, z_1, z_2, \ldots, z_\ell; q,p,t,a,b) \\ & \qquad = \sum_{\nu\prec\lambda} W_{\lambda/\nu}(yt^{-\ell}; q,p,t,at^{2\ell},bt^\ell)\, W_{\nu/\mu}(z_1, \ldots, z_\ell; q,p,t,a,b). \end{aligned} \tag{3.10}$$

The definition of the elliptic Jackson coefficients will be needed below. Let λ and μ be again partitions of at most n–parts such that λ/μ is a skew partition. Then the Jackson coefficients $\omega_{\lambda/\mu}$ are defined by

$$\begin{aligned}\omega_{\lambda/\mu}(x;r,q,p,t;a,b) := {} & \frac{(x^{-1},ax)_\lambda}{(qbx,qb/ax)_\lambda}\frac{(qbr^{-1}x,qb/axr)_\mu}{(x^{-1},ax)_\mu} \\ & \cdot\frac{(r,br^{-1}t^{1-n})_\mu}{(qbr^{-2},qt^{n-1})_\mu}\prod_{i=1}^{n}\left\{\frac{\theta(br^{-1}t^{2-2i}q^{2\mu_i})}{\theta(br^{-1}t^{2-2i})}\left(qt^{2i-2}\right)^{\mu_i}\right\} \\ \cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}}\frac{(br^{-1}t^{3-i-j})_{\mu_i+\mu_j}}{(br^{-1}t^{2-i-j})_{\mu_i+\mu_j}}\right\} & W_\mu(q^\lambda t^{\delta(n)};q,p,t,bt^{2-2n},br^{-1}t^{1-n})\end{aligned} \tag{3.11}$$

where $x,r,q,p,t,a,b\in\mathbb{C}$.

Note that $W_{\lambda/\mu}(x;q,p,t,a,b)$ vanishes unless λ/μ is a horizontal strip, whereas $\omega_{\lambda/\mu}(x;r;a,b)=\omega_{\lambda/\mu}(x;r,q,p,t;a,b)$ is defined even when λ/μ is not a horizontal strip.

The operator characterization [14] of $\omega_{\lambda/\mu}$ yields a recursion formula for Jackson coefficients in the form

$$\omega_{\lambda/\tau}(y,z;r;a,b) := \sum_\mu \omega_{\lambda/\mu}(r^{-k}y;r;ar^{2k},br^k)\,\omega_{\mu/\tau}(z;r;a,b) \tag{3.12}$$

where $y=(x_1,\dots,x_{n-k})\in\mathbb{C}^{n-k}$ and $z=(x_{n-k+1},\dots,x_n)\in\mathbb{C}^k$.

Using the recurrence relation (3.12) the definition of $\omega_{\lambda/\mu}(x;r;a,b)$ can be extended from the single variable $x\in\mathbb{C}$ case to the multivariable function $\omega_{\lambda/\mu}(z;r;a,b)$ with arbitrary number of variables $z=(x_1,\dots,x_n)\in\mathbb{C}^n$. That $\omega_{\lambda/\mu}(z;r;a,b)$ is symmetric is also proved in [14] using a remarkable elliptic BC_n ${}_{10}\varphi_9$ transformation identity.

With these notation and definitions, the $M(a,b)$ and $S(b)$ matrices are now defined.

Definition 3.1. Let λ be a partition of at most n parts and q,t,a,b,ρ and σ be complex parameters. Define the infinite matrix $M(a,b)$ by

$$M_{\lambda\mu}(a,b) := \frac{b^{|\lambda|}}{a^{|\mu|}}\frac{(a/b)_\lambda}{(qb)_\lambda}K_\mu(b)\,W_\mu(q^\lambda t^{\delta(n)};q,p,t,at^{2-2n},bt^{1-n}) \tag{3.13}$$

where

$$\begin{aligned}K_\mu(b)=K_\mu(b,n) := {} & q^{|\mu|}t^{2n(\mu)}\frac{(bt^{1-n})_\mu}{(qt^{n-1})_\mu}\prod_{i=1}^{n}\left\{\frac{\theta(bt^{2-2i}q^{2\mu_i})}{\theta(bt^{2-2i})}\right\} \\ & \cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\right\}\end{aligned} \tag{3.14}$$

and the infinite diagonal matrix $S(b)$ with diagonal entries

$$S_\lambda(b) := \frac{(\sigma,\rho)_\lambda}{(qb/\sigma,qb/\rho)_\lambda}\left(\frac{qb}{\rho\sigma}\right)^{|\lambda|} \tag{3.15}$$

where $|\lambda|=\sum_{i=1}^n\lambda_i$ and $n(\lambda)=\sum_{i=1}^n(i-1)\lambda_i$.

It will be verified that $M(a,b)$ is lower triangular in the sense defined above, and that the n-dependence of definitions (3.13) and (3.15) is not essential.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be an n–part partition with $\lambda_{n-m+1} \neq 0$ and $\lambda_{n-m+1} = \ldots = \lambda_n = 0$ where $1 \leq m \leq n$ and $\hat{\lambda}$ denote the $(n-m)$–part partition obtained by dropping the last m zero parts from λ. That is, $\hat{\lambda} = (\lambda_1, \ldots, \lambda_{n-m})$.

Lemma 3.2. *$M(a,b)$ is lower triangular with respect to partial inclusion ordering, and its entries $M_{\lambda\mu}(a,b)$ are independent of any representation of λ. That is, with the notation as above, $M_{\lambda\mu}(a,b)$ and $M_{\hat{\lambda}\hat{\mu}}(a,b)$ are identical.*

Proof. The fact that $M(a,b)$ is lower triangular follows from the vanishing property [14] of W_μ function. Namely,

$$W_\mu(q^\lambda t^\delta; q, p, t, a, b) = 0 \tag{3.16}$$

when $\mu \not\subseteq \lambda$.

That $M_{\lambda\mu}(a,b)$ is independent of representations of indexing partitions follows from an analogous property [14] of $\omega_{\lambda/\mu}$ written as

$$\omega_{\lambda/\mu}(x; r; a, b) = \omega_{\hat{\lambda}/\hat{\mu}}(x; r; a, b). \tag{3.17}$$

The result follows by noting that

$$\omega_{\lambda\mu}(r^{-1}; r, ar^2, br) = \frac{(ar)_\lambda (qb/ar)_\mu}{(qb/a)_\lambda (ar)_\mu}\, b^{-|\lambda|+|\mu|} r^{|\mu|}\, M_{\lambda\mu}(br, b) \tag{3.18}$$

since factors of the form $u^{|\lambda|}$ and $(u)_\lambda$ are clearly independent of n. □

A key identity in the development of the BC_n Bailey Lemma is now proved. This identity is equivalent to the cocycle identity [14] for $\omega_{\lambda/\mu}$ which can be written in the form

$$\begin{aligned} \omega_{\nu/\mu}((uv)^{-1}; uv; a(uv)^2, buv) \\ = \sum_{\mu \subseteq \lambda \subseteq \nu} \omega_{\nu/\lambda}(v^{-1}; v; a(vu)^2, bvu)\, \omega_{\lambda/\mu}(u^{-1}; u; au^2, bu) \end{aligned} \tag{3.19}$$

for $u, v \in \mathbb{C}$.

Lemma 3.3 (Key Lemma)**.** *With the definitions as above,*

$$S^{-1}(a)\, M(c,a)\, S(a) = S^{-1}(b)\, M(c,b)\, S(b)\, M(b,a) \tag{3.20}$$

where $qab = c\sigma\rho$.

Proof. The proof follows from the observation (3.18) and the cocycle identity (3.19) after a simple reparametrization. □

Two immediate corollaries of this key result are in order.

Corollary 3.4. *For complex parameters a, b and c,*

$$M(c,a) = M(c,b)\, M(b,a) \tag{3.21}$$

Proof. Notice that $S(b) = I$ if $\rho = \sigma = (aq)^{1/2}$ where I is the identity matrix whose entries are $I_{\lambda\mu} = \delta_{\lambda\mu}$. Therefore, setting $\rho = \sigma = (aq)^{1/2}$ in the identity (3.20) yields the identity (3.21) to be proved. □

This result defines a cocycle relation for the matrices $M(a,b)$. The next result shows that the matrices $M(a,b)$ are invertible just as in the classical case [11].

Corollary 3.5. *For $a, b \in \mathbb{C}$,*

$$M(b,b) = I, \quad \text{and} \quad M^{-1}(a,b) = M(b,a) \tag{3.22}$$

Proof. It has been established in [14] that $\omega_{\lambda\mu}(1;1,a,b) = \delta_{\lambda\mu}$. Therefore the identity (3.18) implies that $M(b,b) = I$. It then follows from Corollary 3.4 that $M(b,a)$ and $M(a,b)$ are inverses of each other. □

The $M(a,b)$ matrices satisfy the following elliptic transformation identities.

Lemma 3.6. *With the definitions as above,*

$$M_{\lambda\mu}(pa,b) = (-1)^{|\lambda|} b^{|\lambda|-|\mu|} a^{-|\lambda|} t^{n(\lambda)+2n(\mu)} q^{-|\mu|-n(\lambda')-2n(\mu')} M_{\lambda\mu}(a,b) \tag{3.23}$$

and

$$\begin{aligned} M_{\lambda\mu}(pa,pb) \\ &= b^{-|\lambda|} a^{|\mu|} (-1)^{|\lambda|+|\mu|} p^{|\lambda|-|\mu|} q^{-|\lambda|+|\mu|-n(\lambda')+n(\mu')} t^{n(\lambda)-n(\mu)} M_{\lambda\mu}(a,b) \end{aligned} \tag{3.24}$$

Proof. The proof follows from the observation (3.18) and the following identities given in [14].

$$\omega_{\lambda/\mu}(x;r;a,pb) = (q^2b^2/a)^{|\lambda|-|\mu|} t^{-2n(\lambda)+2n(\mu)} q^{2n(\lambda')-2n(\mu')} \omega_{\lambda/\mu}(x;r;a,b) \tag{3.25}$$

$$\begin{aligned} \omega_{\lambda/\mu}(x;r;pa,b) \\ &= (qb)^{-|\lambda|+|\mu|} p^{|\lambda|-|\mu|} r^{-|\mu|} t^{2n(\lambda)-2n(\mu)} q^{-2n(\lambda')+2n(\mu')} \omega_{\lambda/\mu}(x;r;a,b) \end{aligned} \tag{3.26}$$

$$\omega_{\lambda/\mu}(x;pr;a,b) = (ar^{-2})^{-|\mu|} p^{2|\mu|} t^{2n(\mu)} q^{-2n(\mu')} \omega_{\lambda/\mu}(x;r;a,b) \tag{3.27}$$

where $x \in \mathbb{C}$. □

3.1. **Two Parameter Elliptic Bailey Lemma.** The abstract matrix formulation of the two–parameter Bailey Lemma and consequently the notion of a Bailey lattice was given in one dimension by Agarwal, Andrews and Bressoud [1] and later by Bressoud [12]. Milne and Lilly [27], and Andrews, Schilling and Warnaar [7] extended Bailey Lemma to root systems of type A_n and C_n of rank n and A_2, respectively [42]. This section generalize Bailey Lemma to elliptic level for the non-reduced root system BC_n of rank n, further extending earlier results.

The notion of a Bailey pair will be needed. Let $\mathbb{K}$ be the field of rational functions in $\sigma_i, \rho_i, a_i, b_i \in \mathbb{C}$ for $i \in \mathbb{Z}_>$ over the field $\mathbb{C}(q,p,t)$.

Definition 3.7. The infinite sequences α and β of rational functions $\alpha_\lambda, \beta_\lambda \in \mathbb{K}$ indexed by partitions form a Bailey pair relative to (b_1, a_1) if they satisfy

$$\beta_\lambda = \sum_\mu M_{\lambda\mu}(b_1,a_1)\,\alpha_\mu \tag{3.28}$$

where the sum is over partitions.

The two parameter BC_n Bailey Lemma can now be proved.

Theorem 3.8 (BC_n Bailey Lemma)**.** *With the notation as above, suppose that (α,β) form a Bailey pair relative to (b_1,a_1). Then the pair β' and α' defined by*

$$\beta' = S(a_2)S^{-1}(b_1)M(b_2,b_1)S(b_1)\,\beta \tag{3.29}$$

and

$$\alpha' = S(a_1)M(a_2, a_1)\,\alpha \tag{3.30}$$

form a Bailey pair relative to (b_2, a_2) *provided that* $qa_1b_1 = a_2\rho\sigma$.

Proof. The proof is an immediate consequence of the Lemma 3.3 and the Corollary 3.4. □

Remark 3.9. Note that in the iteration scheme above the parameters σ_i, ρ_i are replaced by σ_{i+1}, ρ_{i+1} and the parameters (a_i, b_i) are replaced by (a_{i+1}, b_{i+1}) in the i–th step.

Remark 3.10. Note also that the elliptic Bailey Lemma of Theorem 3.8 yields a two parameter basic (trigonometric) Bailey Lemma when $p = 0$. In that case $(a)_\lambda$ becomes

$$(a)_\lambda = (a; q, 0, t)_\lambda := \prod_{k=1}^{n} (at^{1-i}; q, 0)_{\lambda_i} = (at^{1-i}; q)_{\lambda_i} \tag{3.31}$$

where the standard q–Pochhammer symbol $(a; q)_n$ is as defined in (1.1). The notation $(a)_\lambda$ will be used for both the elliptic and the basic (trigonometric) case, but the meaning will be clear from the context.

3.2. **One Parameter Basic BC_n Bailey Lemma.** This section and the rest of the paper uses only the basic (trigonometric) case $p = 0$ of BC_n Bailey Lemma. Since the limiting cases of the basic W functions $W_{\lambda/\mu}(x; q, t, a, b) = W_{\lambda/\mu}(x; q, 0, t, a, b)$ will be used in computations, some more notation is needed. Set

$$W_\mu(x; q, t, at^{2-2n}, 0) := \lim_{b\to 0} W_\mu(x; q, t, at^{2-2n}, bt^{1-n}) \tag{3.32}$$

and

$$W_\mu(x; q, t, 0, bt^{1-n}) := \lim_{a\to 0} (a/b)^{|\mu|}\, W_\mu(x; q, t, at^{2-2n}, bt^{1-n}) \tag{3.33}$$

and, finally

$$W_\mu(q^\lambda t^{\delta(n)}; q, t, (u/v)t^{n-1}) := \lim_{d\to 0} W_\mu(q^\lambda t^{\delta(n)}; q, t, dvt^{2-2n}, dut^{1-n}) \tag{3.34}$$

The existence of these limits can be seen from ($p = 0$ case of) the definition (3.9), the recursion formula (3.10) and the limit rule

$$\lim_{a\to 0} a^{|\mu|}(x/a)_\mu = (-1)^{|\mu|}\, x^{|\mu|} t^{-n(\mu)} q^{n(\mu')} \tag{3.35}$$

The limiting cases (3.32), (3.33) and (3.34) are essentially equivalent to known families of symmetric functions such as Okounkov's [32] interpolation Macdonald polynomials P^*_λ, $\bar{P}^*_\lambda$, which in turn generalize Macdonald polynomials P_λ. The exact relationship between these functions are investigated in [16].

Analogous to the matrix formulation of the classical Bailey Lemma [1], the one parameter BC_n Bailey matrix $M(b)$ is also defined as a limiting case of the basic $M(a, b)$ matrix.

Definition 3.11. Let λ be a partition of at most n–parts and $b \in \mathbb{C}$. Define

$$M_{\lambda\mu}(b) := L_\mu(b)\, W_\mu(q^\lambda t^{\delta(n)}; q, t, 0, bt^{1-n}) \tag{3.36}$$

where

$$L_\mu(b) := (-1)^{|\mu|} q^{2|\mu|+n(\mu')} t^{n(\mu)} \frac{(bt^{1-n})_\mu}{(qt^{n-1})_\mu} \cdot \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j} (bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (bt^{2-i-j})_{\mu_i+\mu_j}} \right\} \tag{3.37}$$

and $n(\lambda') = \sum_{i=1}^n \binom{\lambda_i}{2}$.

Remark 3.12. Note that

$$M_{\lambda\mu}(b) = P_\lambda(b)\, M_{\lambda\mu}(0,b)\, Q_\mu(b) \tag{3.38}$$

where

$$P_\lambda(b) = b^{-|\lambda|}(qb)_\lambda \tag{3.39}$$

and

$$Q_\mu(b) = (-1)^{|\mu|} q^{|\mu|+n(\mu')} b^{|\mu|} t^{-n(\mu)} \prod_{i=1}^n \left\{ \frac{(1-bt^{2-2i})}{(1-bt^{2-2i}q^{2\mu_i})} \right\} \tag{3.40}$$

Therefore it follows from the properties of the $M(a,b)$ given in Lemma 3.2 that $M(b)$ is also lower triangular and is independent of representations of λ.

The property that $M_{\lambda\mu}(b)$ satisfies hyperoctahydral symmetry in the rectangular case $\lambda = k^n$ follows from the fact that these matrix entries are well–poised hypergeometric series. The proof of multiple q–series identities below uses the symmetries that are now verified.

Lemma 3.13. *Let $\lambda = k^n$ for some non–negative integer k and set $q^{z_i} = b^{1/2} t^{1-i}$ in the definition (3.36). Then the matrix $M_{k^n\mu}(b)$ is invariant under the standard action $q^{\mu+z} \leftrightarrow q^{w(\mu+z)}$ (permutations and sign changes) of the hyperoctahedral group of rank n.*

Proof. Let $x \in \mathbb{C}$. The following analogue of the Weyl degree formula for W functions

$$\begin{aligned} &W_\mu(xt^{\delta(n)}; q, t, a, b) \\ &\quad = \frac{(x^{-1}, axt^{n-1})_\mu}{(qbxt^{n-1}, qb/(ax))_\mu} \cdot \prod_{1\le i<j\le n} \frac{(t^{j-i+1})_{\mu_i-\mu_j} (qbt^{n-i-j+1})_{\mu_i+\mu_j}}{(t^{j-i})_{\mu_i-\mu_j} (qbt^{n-i-j})_{\mu_i+\mu_j}} \end{aligned} \tag{3.41}$$

follows from the basic (i.e., $p = 0$) version of the W–Jackson sum [14]

$$\begin{aligned} &W_\lambda(s^{-1}x; q, p, t, at^{-2n}s^2, bt^{-n}s) \\ &= \frac{(s, ast^{-n-1})_\lambda}{(qbt^{-1}, qbt^n/a)_\lambda} \cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i+1})_{\lambda_i-\lambda_j} (qbst^{-i-j+1})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j} (qbst^{-i-j})_{\lambda_i+\lambda_j}} \right\} \\ &\quad \cdot \sum_{\mu\subseteq\lambda} \frac{(bt^{-n}, qbt^n/(as))_\mu}{(qt^{n-1}, ast^{-n-1})_\mu} \cdot \prod_{i=1}^n \left\{ \frac{\theta(bt^{1-2i}q^{2\mu_i})}{\theta(bt^{1-2i})} (qt^{2i-2})^{\mu_i} \right\} \\ &\cdot \prod_{1\le i<j\le n} \left\{ \frac{(t^{j-i})_{\mu_i-\mu_j} (qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j} (t^{j-i+1})_{\mu_i-\mu_j}} \frac{(bqt^{-i-j})_{\mu_i+\mu_j} (bt^{-i-j+2})_{\mu_i+\mu_j}}{(bt^{-i-j+1})_{\mu_i+\mu_j} (qbt^{-i-j+1})_{\mu_i+\mu_j}} \right\} \\ &\quad \cdot W_\mu(q^\lambda t^{\delta(n)}; q, p, t, bst^{1-2n}, bt^{-n}) \cdot W_\mu(x; q, p, t, at^{-2n}, bt^{-n}) \end{aligned} \tag{3.42}$$

and the fact [14] that

$$W_\mu(t^{\delta(n)};q,t,a,b)=\delta_{0\mu}. \tag{3.43}$$

Set $\lambda=k^n$ and $q^{z_i}=b^{1/2}t^{1-i}$ in (3.36). Using the identity (3.41), the definition of the q–Pochhammer symbol (1.1), and the identities

$$(v)_n=\frac{(-v)^n q^{\binom{n}{2}}}{(qv^{-1})_{-n}} \tag{3.44}$$

and

$$\prod_{1\le i<j\le n}\frac{a_i}{a_j}=\prod_{i=1}^n a_i^{1+n-2i} \tag{3.45}$$

the entries of the matrix $M(b)$ can be written in the form

$$\begin{aligned} M_{k^n\mu}(b)=&\prod_{i=1}^n\frac{(q^{-k}b^{-1/2}q^{-z_i})_\infty(q^{-k}b^{-1/2}q^{z_i})_\infty}{(qb^{-1/2}t^{n-1}q^{-z_i})_\infty(qb^{-1/2}t^{n-1}q^{z_i})_\infty}\\ &\cdot\prod_{1\le i<j\le n}\left\{\frac{(q^{1-z_j+z_i})_\infty(q^{1+z_j-z_i})_\infty}{(qt^{-1}q^{z_i-z_j})_\infty(qt^{-1}q^{-z_i+z_j})_\infty}\frac{(q^{1-z_i-z_j})_\infty(q^{1+z_i+z_j})_\infty}{(t^{-1}q^{1-z_i-z_j})_\infty(t^{-1}q^{1+z_i+z_j})_\infty}\right\}\\ &\cdot\prod_{i=1}^n\frac{(qb^{-1/2}t^{n-1}q^{-\mu_i-z_i})_\infty(qb^{-1/2}t^{n-1}q^{\mu_i+z_i})_\infty}{(q^{-k}b^{-1/2}q^{-\mu_i-z_i})_\infty(q^{-k}b^{-1/2}q^{\mu_i+z_i})_\infty}\\ &\cdot\prod_{1\le i<j\le n}\left\{\frac{(q^{1+\mu_i+z_i-\mu_j-z_j})_\infty(q^{1-\mu_i-z_i+\mu_j+z_j})_\infty}{(t^{-1}q^{1+\mu_i+z_i-\mu_j-z_j})_\infty(t^{-1}q^{1-\mu_i-z_i+\mu_j+z_j})_\infty}\right.\\ &\left.\cdot\frac{(q^{1-\mu_i-z_i-\mu_j-z_j})_\infty(q^{1+\mu_i+z_i+\mu_j+z_j})_\infty}{(t^{-1}q^{1-\mu_i-z_i-\mu_j-z_j})_\infty(t^{-1}q^{1+\mu_i+z_i+\mu_j+z_j})_\infty}\right\} \end{aligned} \tag{3.46}$$

It is now clear that $M_{k^n\mu}(b)$ is invariant under the maps $q^{\mu_i+z_i}\leftrightarrow q^{\pm(\mu_j+z_j)}$. □

It could be shown, using the definition and the recurrence relation for W_λ, that $M_{\lambda\mu}(b)$ satisfies hyperoctahedral group symmetry for a general partition λ. However, the rectangular case given in Lemma 3.13 will be sufficient for the proof of the multiple q–series identities obtained below.

The diagonal $S(b)$ matrix has similar properties.

Lemma 3.14. *The entries of the basic $S(b)$ matrix (3.15) is independent of representations of λ and satisfies the hyperoctahedral group symmetry.*

Proof. That $S_\lambda(b)$ is independent of different representations of λ is obvious. Setting $q^{z_i}=b^{1/2}t^{1-i}$ in (3.15) and using the identities given above, one obtains

$$\begin{aligned} S_\lambda(b)=&\prod_{i=1}^n\frac{(\sigma b^{-1/2}q^{z_i})_\infty(\sigma b^{-1/2}q^{-z_i})_\infty}{(q\rho^{-1}b^{1/2}q^{z_i})_\infty(q\rho^{-1}b^{1/2}q^{-z_i})_\infty}\\ &\cdot\frac{(q\rho^{-1}b^{1/2}q^{z_i+\lambda_i})_\infty(q\rho^{-1}b^{1/2}q^{-(z_i+\lambda_i)})_\infty}{(\sigma b^{-1/2}q^{z_i+\lambda_i})_\infty(\sigma b^{-1/2}q^{-(z_i+\lambda_i)})_\infty} \end{aligned} \tag{3.47}$$

proving the invariance under the maps $q^{\lambda+z}\leftrightarrow q^{w(\lambda+z)}$ or $q^{\lambda_i+z_i}\leftrightarrow q^{\pm(\lambda_i+z_i)}$ for all $i\in[n]$. □

The fact that $M(b)$ is invertible is a simple consequence of the Corollary 3.4.

Lemma 3.15. *With the notation as above, the inverse $M^{-1}(b)$ of the infinite triangular matrix $M(b)$ is given by*

$$M^{-1}_{\lambda\mu}(b) = \frac{q^{-|\lambda|+|\mu|}t^{2n(\mu)}}{(qb, qt^{n-1})_\mu} \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})} \right\} \cdot \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \right\} \cdot W_\mu(q^\lambda t^{\delta(n)}; q, t, bt^{2-2n}, 0) \tag{3.48}$$

Proof. Setting $c = a$ in Lemma (3.4) gives the identity $I = M(a, b)\, M(b, a)$ by the virtue of (3.22). This beautiful identity can be written explicitly in the form

$$\delta_{\lambda\tau} = \sum_\mu \frac{b^{|\lambda|}}{a^{|\mu|}} \frac{(a/b)_\lambda}{(qb)_\lambda} K_\mu(b)\, W_\mu(q^\lambda t^{\delta(n)}; q, t, at^{2-2n}, bt^{1-n}) \cdot \frac{a^{|\mu|}}{b^{|\tau|}} \frac{(b/a)_\mu}{(qa)_\mu} K_\tau(a)\, W_\tau(q^\mu t^{\delta(n)}; q, t, bt^{2-2n}, at^{1-n}) \tag{3.49}$$

Noting that

$$K_\mu(b) = (-1)^{|\mu|} q^{-|\mu|-n(\mu')} t^{n(\mu)} \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})} \right\} L_\mu(b) \tag{3.50}$$

and taking the limit $a \to 0$ of (3.49) by using (3.35), one gets the inverse of $M(b)$ given above. □

Next, the closed form of the entries of the matrix $N(b) := M(b)\, S(b)\, M^{-1}(b)$ will be computed. In the classical one dimensional case, $N(b)$ is computed by means of the Bailey Transform and the Saalchutz's ${}_3\varphi_2$ summation formula. Instead of generalizing this idea, the two parameter basic BC_n Bailey Lemma is used to prove the next result.

Lemma 3.16. *Let $q, t, b, \sigma, \rho \in \mathbb{C}$. Then the entries of the matrix $N(b)$ is given in the closed form*

$$N_{\lambda\mu}(b) = q^{|\mu|} t^{2n(\mu)} \frac{(qb)_\lambda (qb/\rho\sigma)_\lambda}{(qb/\sigma)_\lambda (qb/\rho)_\lambda} \frac{(\sigma)_\mu (\rho)_\mu}{(qb)_\mu (qt^{n-1})_\mu} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}} \right\} \cdot W_\mu(q^\lambda t^{\delta(n)}; q, t, qbt^{n-1}/\rho\sigma) \tag{3.51}$$

Proof. Note that one can write

$$M_{\lambda\mu}(b) = P_\lambda(b)\, M_{\lambda\mu}(0, b)\, Q_\mu(b) \tag{3.52}$$

where

$$P_\lambda(b) = b^{-|\lambda|}(qb)_\lambda \tag{3.53}$$

and

$$Q_\mu(b) = (-1)^{|\mu|} q^{|\mu|+n(\mu')} b^{|\mu|} t^{-n(\mu)} \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i})}{(1-bt^{2-2i}q^{2\mu_i})} \right\} \tag{3.54}$$

Set $a = du$ and $c = dv$ for some $d, u, v \in \mathbb{C}$ in (3.20). Due to the relation $qub = v\rho\sigma$ one gets $v/u = qb/\rho\sigma$. Rewrite Lemma (3.3) in the form

$$M(dv, b)\, S(b)\, M(b, du) = S(b)\, S^{-1}(du)\, M(dv, du)\, S(du) \tag{3.55}$$

Now in the limit as $d \to 0$, the left hand side becomes

$$\lim_{d\to 0} M(dv,b)\,S(b)\,M(b,du) = M(0,b)\,S(b)\,M(b,0) \tag{3.56}$$

which is essentially equal to $N(b)$ up to diagonal factors. That is,

$$N(b) = \lim_{d\to 0} \left\{P\,S(b)\,S^{-1}(du)\,M(dv,du)\,S(du)\,P^{-1}\right\} \tag{3.57}$$

which gives the closed form to be computed. □

The one parameter basic BC_n Bailey Lemma is proved next. The notion of a Bailey pair translates to the one parameter case in an obvious way. Namely, the infinite sequences α and β of rational functions $\alpha_\lambda, \beta_\lambda \in \mathbb{K}$ indexed by partitions form a Bailey pair (relative to $b = b_1$) if they satisfy $\beta_\lambda = \sum_\mu M_{\lambda\mu}(b)\,\alpha_\mu$ where the sum is over partitions.

Theorem 3.17. *Suppose that the infinite sequences α and β form a Bailey pair relative to b. Then α' and β' also form a Bailey pair relative to b where*

$$\alpha'_\lambda = S_\lambda(b)\,\alpha_\lambda \tag{3.58}$$

and

$$\beta'_\lambda = \sum_\mu N_{\lambda\mu}(b)\,\beta_\mu \tag{3.59}$$

where the sum is over partitions.

Proof. The proof is a consequence of the definition of $N(b)$ and that of a Bailey pair relative to b. □

3.3. **Generalized Watson transformation.** The power of Bailey Lemma comes from its potential for iteration. The lemma can be applied to a given Bailey pair (α,β) repeatedly producing an infinite sequence of Bailey pairs $(\alpha,\beta) \to (\alpha',\beta') \to (\alpha'',\beta'') \to \cdots$, what is called a Bailey chain. In fact, a stronger result says that it is possible to walk along the Bailey chain in every direction as depicted in the following figure.

$$\begin{array}{ccccccccccccc}
\cdots & \leftrightarrow & \alpha^{(-2)} & \leftrightarrow & \alpha^{(-1)} & \leftrightarrow & \alpha & \leftrightarrow & \alpha' & \leftrightarrow & \alpha'' & \leftrightarrow & \cdots \\
 & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow & & \\
\cdots & \leftrightarrow & \beta^{(-2)} & \leftrightarrow & \beta^{(-1)} & \leftrightarrow & \beta & \leftrightarrow & \beta' & \leftrightarrow & \beta'' & \leftrightarrow & \cdots
\end{array}$$

FIGURE 1. Bailey chain

Lemma 3.18 (Bailey Walk)**.** *The entire Bailey chain is uniquely determined by a single node $\alpha^{(i)}$ or $\beta^{(i)}$ for any $i \in \mathbb{Z}$ on the chain.*

Proof. The lower triangular matrices $M(b)$, $N(b)$ and the diagonal matrix $S(b)$, having no zero entries on their diagonal, are all invertible. One move forward and backward in the first line in Figure 1 by $S(b)$ and $S^{-1}(b)$, in the second line by $N(b)$ and $N^{-1}(b)$ and move up and down by $M(b)$ and $M^{-1}(b)$. $M^{-1}(b)$ is obtained before in Lemma (3.15), and the fact that $N(b)$ is invertible follows from its definition. □

This powerful iteration mechanism allows one to prove numerous multiple basic hypergeometric series and multiple q-series identities. In this section a terminating ${}_6\varphi_5$ summation formula and a generalized Watson transformation will be given. The limiting cases of these results are used to prove Euler's Pentagonal Number Theorem, the Rogers–Ramanujan identities and the extreme cases of the Andrews–Gordon identities. The details for the Rogers–Ramanujan identities will be given in this paper and other results will appear in future publications [15], [16].

The Bailey pair (α, β) corresponding to the simplest non–trivial sequence β defined by $\beta_\lambda = \delta_{\lambda 0}$ is called the unit Bailey pair. The corresponding α sequence can easily be computed using the the inverse matrix $M^{-1}(b)$. One gets

$$\alpha_\lambda = \sum_\mu M^{-1}_{\lambda\mu}(b)\,\beta_\mu = q^{-|\lambda|}\prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})}\right\} \tag{3.60}$$

Iterating the Bailey Lemma of Theorem 3.17 once, that is computing the sequences α' and β' and writing out the relation $\beta'_\lambda = \sum_\mu M_{\lambda\mu}(b)\,\alpha'_\mu$ explicitly gives a higher dimensional analogue of the terminating ${}_6\varphi_5$ summation formula

$$\begin{aligned}\frac{(qb, qb/\rho_1\sigma_1)_\lambda}{(qb/\sigma_1, qb/\rho_1)_\lambda} &= \sum_{\mu\subseteq\lambda}(-1)^{|\mu|}q^{n(\mu')}t^{n(\mu)}\left(\frac{q^2b}{\sigma_1\rho_1}\right)^{|\mu|}\frac{(bt^{1-n},\sigma_1,\rho_1)_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1)_\mu}\\ &\cdot\prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\right\}\\ &\cdot W_\mu(q^\lambda t^{\delta(n)};q,t,0,bt^{1-n})\end{aligned} \tag{3.61}$$

A second iteration of Bailey Lemma and consequently writing up the relation $\beta''_\lambda = \sum_\mu M_{\lambda\mu}(b)\,\alpha''_\mu$ gives

$$\begin{aligned}&\frac{(qb, qb/\rho_2\sigma_2)_\lambda}{(qb/\sigma_2, qb/\rho_2)_\lambda}\sum_{\mu\subseteq\lambda}q^{|\mu|}t^{2n(\mu)}\frac{(\sigma_2,\rho_2,qb/\rho_1\sigma_1)_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1)_\mu}\\ &\quad\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}}\right\}W_\mu(q^\lambda t^{\delta(n)};q,t,qbt^{n-1}/\rho_2\sigma_2)\\ &= \sum_{\mu\subseteq\lambda}\left(\frac{q^3b^2}{\sigma_1\rho_1\sigma_2\rho_2}\right)^{|\mu|}(-1)^{|\mu|}q^{n(\mu')}t^{n(\mu)}\prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}\\ &\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\right\}W_\mu(q^\lambda t^{\delta(n)};q,t,0,bt^{1-n})\\ &\cdot\frac{(bt^{1-n},\sigma_2,\rho_2,\sigma_1,\rho_1)_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1,qb/\sigma_2,qb/\rho_2)_\mu}\end{aligned} \tag{3.62}$$

This is a higher dimensional analogue of the Watson transformation (2.4).

Iterating the Bailey Lemma of Theorem 3.17 N times yields an extension of the Watson transformation (3.62) which is called the generalized Watson transformation. The following notation will be used.

$$\sum_{\nu_n\subseteq\nu_{n-1}\subseteq\cdots\subseteq\nu_1\subseteq\nu_0} := \sum_{\nu_1\subseteq\nu_0}\sum_{\nu_2\subseteq\nu_1}\cdots\sum_{\nu_n\subseteq\nu_{n-1}} \tag{3.63}$$

Lemma 3.19. *With the notation as above,*

$$\sum_{\mu^{N-1}\subseteq\cdots\subseteq\mu^0}\prod_{k=1}^{N-1}\Bigg\{q^{|\mu^k|}t^{2n(\mu^k)}\frac{(qb/\rho_{N-k}\sigma_{N-k},\sigma_{N-k+1},\rho_{N-k+1})_{\mu^k}}{(qb/\sigma_{N-k},qb/\rho_{N-k},qt^{n-1})_{\mu^k}}$$
$$\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i^k-\mu_j^k}}{(qt^{j-i-1})_{\mu_i^k-\mu_j^k}}\right\}W_{\mu^k}(q^{\mu^{k-1}}t^{\delta(n)};q,t,qbt^{n-1}/\rho_{N-k+1}\sigma_{N-k+1})\Bigg\}$$
$$=\frac{(qb/\sigma_N,qb/\rho_N)_{\mu^0}}{(qb,qb/\rho_N\sigma_N)_{\mu^0}}\cdot\sum_{\mu\subseteq\mu^0}(-1)^{|\mu|}q^{|\mu|+n(\mu')}t^{n(\mu)}\frac{(bt^{1-n})_\mu}{(qt^{n-1})_\mu}\prod_{i=1}^n\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}$$
$$\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\right\}W_\mu(q^{\mu^0}t^{\delta(n)};q,t,0,bt^{1-n})$$
$$\cdot\prod_{k=1}^N\left\{\frac{(\sigma_{N-k+1},\rho_{N-k+1})_\mu}{(qb/\sigma_{N-k+1},qb/\rho_{N-k+1})_\mu}\left(\frac{qb}{\sigma_{N-k+1}\rho_{N-k+1}}\right)^{|\mu|}\right\} \tag{3.64}$$

Proof. Iterate the Bailey Lemma of Theorem 3.17 N times starting with the unit Bailey pair corresponding to $\beta_\lambda=\delta_{\lambda 0}$. □

Note that the $N=1$ case of the generalized Watson transformation (3.64) reduces to the terminating ${}_6\varphi_5$ summation formula (3.61), and the $N=2$ case of it reduces to the Watson transformation (3.62).

Remark 3.20. It has been already seen above in Lemma 3.13 and Lemma 3.14 that the matrices $M(b)$ and $S(b)$ involved in the iteration process are invariant under the action of the hyperoctahedral group, and are independent of different representations of λ. Therefore, the right hand side of the identity (3.64) obtained in the iteration process will always share the same properties when α_λ has the same properties. But, if one sets $q^{z_i}=b^{1/2}t^{1-i}$ in (3.60), α_λ can be written in the form

$$\alpha_\lambda=\prod_{i=1}^n\left\{q^{-2(z_i+\lambda_i)^2+2z_i^2}\frac{(q^{1+2z_i})_\infty}{(q^{1+2(z_i+\lambda_i)})_\infty}\frac{(q^{1-2z_i})_\infty}{(q^{1-2(z_i+\lambda_i)})_\infty}\right\} \tag{3.65}$$

which clearly satisfies these properties. To make the symmetries more transparent, one can rewrite the series in the right hand side of the generalized Watson transformation (3.64) similar to (3.46) by setting $\mu^0=k^n$ and $q^{z_i}=b^{1/2}t^{1-i}$ and using the same definitions and identities, namely (1.1), (3.44), and (3.45) as follows.

$$
\begin{aligned}
(3.66)\quad & \prod_{i=1}^{n} q^{-(N-1)z_i^2} \prod_{i=1}^{n} \frac{(q^{1+2z_i})_\infty (q^{1-2z_i})_\infty}{(qt^{n-i})_\infty (q^{1-2z_i}t^{n-i})_\infty} \\
& \cdot \prod_{1\le i<j\le n} \left\{ \frac{(q^{1+z_i-z_j})_\infty (q^{1-z_i+z_j})_\infty}{(t^{-1}q^{1+z_i-z_j})_\infty (t^{-1}q^{1-z_i+z_j})_\infty} \frac{(q^{1-z_i-z_j})_\infty (q^{1+z_i+z_j})_\infty}{(t^{-1}q^{1-z_i-z_j})_\infty (t^{-1}q^{1+z_i+z_j})_\infty} \right\} \\
& \cdot \sum_{\mu,\ell(\mu)\le n} \prod_{i=1}^{n} q^{(N-1)(\mu_i+z_i)^2} \prod_{i=1}^{n} \frac{(q^{1-z_i+(\mu_i+z_i)}t^{n-i})_\infty (q^{1-z_i-(\mu_i+z_i)}t^{n-i})_\infty}{(q^{1+2z_i+2\mu_i})_\infty (q^{1-2z_i-2\mu_i})_\infty} \\
& \cdot \prod_{1\le i<j\le n} \left\{ \frac{(q^{1+z_i-z_j+\mu_i-\mu_j})_\infty (q^{1-z_i+z_j-\mu_i+\mu_j})_\infty}{(t^{-1}q^{1+z_i-z_j+\mu_i-\mu_j})_\infty (t^{-1}q^{1-z_i+z_j-\mu_i+\mu_j})_\infty} \right. \\
& \left. \cdot \frac{(q^{1-z_i-z_j-\mu_i-\mu_j})_\infty (q^{1+z_i+z_j+\mu_i+\mu_j})_\infty}{(t^{-1}q^{1-z_i-z_j-\mu_i-\mu_j})_\infty (t^{-1}q^{1+z_i+z_j+\mu_i+\mu_j})_\infty} \right\}
\end{aligned}
$$

4. Multiple q–Series Identities Associated to Root Systems

The generalized Watson transformation (3.64) produces, in the limit, several remarkable multiple q–series identities. The initial $N = 1$ instance of the iteration corresponds to the Euler's Pentagonal Number Theorem, and the $N = 2$ instance yields the Rogers–Ramanujan identities. Furthermore, the general N case of (3.64) is enough to prove the Andrews–Gordon identities in the extreme cases [14]. However the full Andrews–Gordon identities requires the two parameter BC_n Bailey Lemma of Theorem 3.8. As noted above, only the details for the Rogers–Ramanujan identities will be given in this paper.

4.1. The Rogers–Selberg Identity.

First, a BC_n analogue of the Rogers–Selberg identity (2.5) will be proved.

Lemma 4.1. *Let $q, t, b \in \mathbb{C}$ and $|q| < 1$. A BC_n generalization of the Rogers–Selberg identity is given by*

$$
\begin{aligned}
(4.1)\quad & \sum_{\ell(\lambda)\le n} (-1)^{|\lambda|} b^{2|\lambda|} t^{(1-n)|\lambda|-3n(\lambda)} q^{2|\lambda|+5n(\lambda')} \frac{(bt^{1-n})_\lambda}{(qt^{n-1})_\lambda} \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})} \right\} \\
& \cdot \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(bt^{3-i-j})_{\lambda_i+\lambda_j}}{(bt^{2-i-j})_{\lambda_i+\lambda_j}} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j} (qbt^{2-i-j})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j} (qbt^{1-i-j})_{\lambda_i+\lambda_j}} \right\} \\
& = (qb)_{\infty^n} \sum_{\ell(\lambda)\le n} \frac{b^{|\lambda|} q^{|\lambda|+2n(\lambda')} t^{(1-n)|\lambda|}}{(qt^{n-1})_\lambda} \prod_{1\le i<j\le n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}} \right\}
\end{aligned}
$$

where $(u)_{\infty^n}$ denotes the product $\prod_{i=1}^{n} (ut^{1-i})_\infty$.

Proof. Set $\lambda = k^n$, for some $k \in \mathbb{Z}_{\geq}$, in the Watson transformation (3.62) and send the parameters $\sigma_1, \rho_1, \sigma_2$ and ρ_2 to ∞ using the limit rule (3.35) to get

$$\begin{aligned}
(4.2)\quad & \frac{1}{(qb)_{k^n}} \sum_{\lambda \subseteq k^n} t^{(1-n)|\lambda|-2n(\lambda)}\, b^{2|\lambda|}\, q^{(2+k)|\lambda|+4n(\lambda')} \\
&\quad \cdot \prod_{i=1}^{n} \left\{ \frac{(1-bt^{2-2i}q^{2\lambda_i})}{(1-bt^{2-2i})} \right\} \frac{(bt^{1-n})_\lambda}{(qt^{n-1})_\lambda} \frac{(q^{-k})_\lambda}{(q^{1+k}b)_\lambda} \\
&\cdot \prod_{1\leq i<j\leq n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(bt^{3-i-j})_{\lambda_i+\lambda_j}}{(bt^{2-i-j})_{\lambda_i+\lambda_j}} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}(qbt^{2-i-j})_{\lambda_i+\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}(qbt^{1-i-j})_{\lambda_i+\lambda_j}} \right\} \\
&\quad = \sum_{\lambda \subseteq k^n} q^{(k+1)|\lambda|+n(\lambda')}\, (-1)^{|\lambda|}\, b^{|\lambda|}\, t^{(1-n)|\lambda|+n(\lambda)} \frac{(q^{-k})_\lambda}{(qt^{n-1})_\lambda} \\
&\qquad \cdot \prod_{1\leq i<j\leq n} \left\{ \frac{(qt^{j-i})_{\lambda_i-\lambda_j}}{(qt^{j-i-1})_{\lambda_i-\lambda_j}} \frac{(t^{j-i+1})_{\lambda_i-\lambda_j}}{(t^{j-i})_{\lambda_i-\lambda_j}} \right\}
\end{aligned}$$

Before passing the limit $k \to \infty$, it would be useful to rewrite this limiting case of the Watson transformation (4.2) in order to verify convergence. Note that factors in the series (4.2) may be flipped using

$$(4.3)\qquad (v;q,t)_\mu = \frac{1}{(qt^{n-1}/v;q,t)_{-\mu^{(r)}}} (-1)^{|\mu|} v^{|\mu|} q^{n(\mu')}\, t^{-n(\mu)}$$

where $-\mu^{(r)}$ is defined to be

$$(4.4)\qquad -\mu^{(r)} := (-\mu_n, \ldots, -\mu_{n+1-i}, \ldots, -\mu_1)$$

Set $q^{z_i} = b^{1/2}t^{1-i}$ in (4.2), flip appropriate factors and use the definition of the q–Pochhammer symbol (1.1) to write the well–poised side (i.e., left hand side) and the balanced side (i.e., right hand side) of (4.2) in the form

$$\begin{aligned}
(4.5)\quad & \prod_{i=1}^{n} \frac{(q^{2z_i}t^{i-n})_\infty}{(qt^{n-i})_\infty} \frac{(q^{1+2z_i})_\infty}{(q^{2z_i})_\infty} \frac{1}{(q^{1+k}t^{i-1})_\infty (q^{1+k+2z_i}t^{i-1})_\infty} \\
&\cdot \prod_{1\leq i<j\leq n} \left\{ \frac{(q^{1+z_i-z_j})_\infty}{(qt^{-1}q^{z_i-z_j})_\infty} \frac{(tq^{z_i+z_j})_\infty}{(q^{z_i+z_j})_\infty} \frac{(tq^{z_i-z_j})_\infty (q^{1+z_i+z_j})_\infty}{(q^{z_i-z_j})_\infty (qt^{-1}q^{z_i+z_j})_\infty} \right\} \\
&\cdot \sum_{\lambda\subseteq k^n} \prod_{i=1}^{n} (-1)^{\lambda_i}\, q^{4z_i\lambda_i} t^{(i-n)\lambda_i} q^{5\lambda_i^2/2-\lambda_i/2} \frac{(t^{n-i}q^{1+\lambda_i})_\infty}{(q^{2z_i}t^{i-n}q^{\lambda_i})_\infty} \frac{(q^{2z_i+2\lambda_i})_\infty}{(q^{1+2z_i+2\lambda_i})_\infty} \\
&\qquad \cdot (q^{1+k}t^{i-1}q^{-\lambda_i})_\infty (q^{1+k}t^{i-1}q^{2z_i+\lambda_i})_\infty \\
&\quad \cdot \prod_{1\leq i<j\leq n} \left\{ \frac{(t^{-1}q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty}{(q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty} \frac{(q^{z_i+z_j+\lambda_i+\lambda_j})_\infty}{(tq^{z_i+z_j+\lambda_i+\lambda_j})_\infty} \right. \\
&\qquad\qquad \left. \cdot \frac{(q^{z_i-z_j+\lambda_i-\lambda_j})_\infty (t^{-1}q^{1+z_i+z_j+\lambda_i+\lambda_j})_\infty}{(tq^{z_i-z_j+\lambda_i-\lambda_j})_\infty (q^{1+z_i+z_j+\lambda_i+\lambda_j})_\infty} \right\}
\end{aligned}$$

and

$$\prod_{1\le i<j\le n}\left\{\frac{(q^{1+z_i-z_j})_\infty}{(qt^{-1}q^{z_i-z_j})_\infty}\frac{(tq^{z_i-z_j})_\infty}{(q^{z_i-z_j})_\infty}\right\}\prod_{i=1}^{n}\frac{(q^{1+2z_i}t^{i-1})_\infty}{(q^{1+k}t^{i-1})_\infty(qt^{n-i})_\infty}$$
$$\cdot\sum_{\lambda\subseteq k^n}\prod_{i=1}^{n}q^{2z_i\lambda_i}t^{2(i-1)\lambda_i}q^{\lambda_i^2}t^{(1-n)\lambda_i}(q^{1+k}t^{i-1}q^{-\lambda_i})_\infty(qt^{n-i}q^{\lambda_i})_\infty$$
$$\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{-1}q^{z_i-z_j+\lambda_i-\lambda_j})_\infty}{(q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty}\frac{(q^{z_i-z_j+\lambda_i-\lambda_j})_\infty}{(tq^{z_i-z_j+\lambda_i-\lambda_j})_\infty}\right\} \tag{4.6}$$

respectively. The Dominated Convergence Theorem will now be applied on both series above as $k\to\infty$ to get the unspecialized Rogers–Selberg identity.

Let $f_\lambda(k)$ denote the summand of the series in (4.5) and $g_\lambda(k)$ denote that in (4.6). Let h be either f or g, and consider the sum

$$\sum_{\lambda\in L_k^+}h_\lambda(k) \tag{4.7}$$

where $L_k^+=\{\lambda\in\mathbb{Z}^n : k\ge\lambda_1\ge\dots\lambda_n\ge 0\}$. Note that, in the limit, the lattice $L^+:=\lim_{k\to\infty}L_k^+$ consists of all partitions of length at most n. First verify that the pointwise limit $h_\lambda:=\lim_{k\to\infty}h_\lambda(k)$ exists for all $\lambda\in L_k^+$. Then compute m_λ^h for each λ such that $|h_\lambda(k)|\le m_\lambda^h$ for any k larger than λ_1, and finally verify that the series $\sum_{\lambda\in L^+}m_\lambda^h$ is convergent.

That the pointwise limit exists on both sides is clear, since

$$\lim_{k\to\infty}(q^{1+k}t^{i-1}q^{-\lambda_i})_\infty(q^{1+k}t^{i-1}q^{2z_i+\lambda_i})_\infty=1 \tag{4.8}$$

on the well–poised side, and

$$\lim_{k\to\infty}(q^{1+k}t^{i-1}q^{-\lambda_i})_\infty=1 \tag{4.9}$$

on the balanced side, and none of the other factors of $f_\lambda(k)$ or $g_\lambda(k)$ depend on k.

Standard theorems on infinite products and sequences imply that all the factors of the form $(uq^\alpha)_\infty/(vq^\alpha)_\infty$ inside the sum are bounded when α is a non–negative integer and $u,v\in\mathbb{C}$ such that the denominator never vanishes, since

$$\lim_{\alpha\to\infty}\left|\frac{(uq^\alpha)_\infty}{(vq^\alpha)_\infty}\right|=1 \tag{4.10}$$

when $|q|<1$. Therefore it follows that for some constants C_f and C_g that depend only on q and z, and are independent of k and λ

$$m_\lambda^f=C_f\left|\prod_{i=1}^{n}(-1)^{\lambda_i}\,q^{4z_i\lambda_i}t^{(i-n)\lambda_i}q^{5\lambda_i^2/2-\lambda_i/2}\right| \tag{4.11}$$

and

$$m_\lambda^g=C_g\left|\prod_{i=1}^{n}q^{2z_i\lambda_i}t^{2(i-1)\lambda_i}q^{\lambda_i^2}t^{(1-n)\lambda_i}\right| \tag{4.12}$$

Finally, it needs to be shown that $\sum_{\lambda\in L^+}m_\lambda^h$ is convergent. But this is clearly true for any $t,b\in\mathbb{C}$ when $|q|<1$ due to the quadratic factors of q. Consider, for example, the multiple series ratio test. Let $\varepsilon_i=(0\dots1\dots0)$ denote the n–tuple of integers with a 1 only in the i–th position and zeroes at other positions. Then one sees that

$$\left|\frac{m^h_{\lambda+\varepsilon_i}}{m^h_\lambda}\right| \tag{4.13}$$

is a function of $|q|^{\lambda_i}$ which becomes arbitrarily small as $\lambda_i \to \infty$ for each $i \in [n]$ where possible (i.e., when $\lambda + \varepsilon_i$ is a partition). Therefore $\sum_{\lambda\in L^+} m^h_\lambda$ converges when $|q| < 1$ as desired. □

Remark 4.2. The BC_n Rogers–Selberg identity (4.1) was given in [13] as a limiting case of the BC_n ${}_{10}\varphi_9$ transformation which was first proved there (also see [14] and [33]).

4.2. **Specializations.** Next, it will be shown that the series on both sides of the Rogers–Selberg identity (4.1) can be multilateralized, that is they can be replaced by series over the full lattice $\mathbb{Z}^n$ under certain specializations of the parameters b and t. An auxiliary result called multilateralization lemma will be needed.

The root system terminology used in the sequel should be introduced at this point [25], [31]. Let R be the root system C_n in the n–dimensional Euclidean space E^n endowed with the standard inner product $\langle\cdot,\cdot\rangle$. Let R^+ denote the set of all positive roots, and W denote the Weyl group of R. The weight lattice

$$L = L(R) := \{\lambda \in E^n : \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}, \forall\, \alpha \in R\} \tag{4.14}$$

and the cone of dominant weights

$$L^+ = L(R^+) := \{\lambda \in E^n : \langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}_\geq, \forall\, \alpha \in R^+\} \tag{4.15}$$

are defined in the standard way. For an arbitrarily fixed $z \in \mathbb{R}^n$, let R_z denote the isomorphic root system obtained by translating R by $-z$ so that the origin is moved to $-z$. Let R^+_z, C_z denote the translated positive roots and fundamental chamber corresponding to R^+ and C of the original root system.

Now, fix a $z \in \mathbb{R}^n$ such that under the reflections w_{α_z} the lattice L remains fixed. This, of course, implies that the standard action of the Weyl group on the translated root system R_z, by permuting and changing signs of the coordinates, leaves the lattice L invariant. In other words, it is required that for all $\mu \in L$ there exists a unique $\lambda \in L$ such that

$$\mu = w(z + \lambda) - z \tag{4.16}$$

for all $w \in W$. This amounts to saying that $w(z) - z \in L$.

Consider the multiple sum over the dominant cone

$$\sum_{\mu\in L_+} f(z + \mu) \tag{4.17}$$

and assume that the summand has the symmetry $f(\mu + z) = f(w(\mu + z))$ for all $w \in W$. The interesting cases are when the sum (4.17) is a (convergent) multiple basic hypergeometric series or a multiple q–series. A technique will next be developed showing how to multilateralize (4.17), that is how to replace it, under certain restrictions, with a series over the entire weight lattice

$$\sum_{\mu\in L} g(z + \mu) \tag{4.18}$$

where g is related to f in a way made precise by Lemma 4.3 below.

Recall that for a reduced root system R Macdonald's polynomial identity [29] can, in the notation given above, be written as

$$\sum_{w\in W}\prod_{\alpha\in R^+}\frac{1-u_\alpha e^{-w\alpha}}{1-e^{-w\alpha}}=\sum_{w\in W}\prod_{\alpha\in R(w)}u_\alpha \tag{4.19}$$

where $R(w)=R^+\cap -wR^+$, u_α are indeterminates indexed by positive roots $\alpha\in R^+$, and e^α are formal exponentials denoting elements in the group ring of the root lattice generated by R. The identity will be rewritten in a more convenient form for the purpose of this paper as

$$\sum_{w\in W}\prod_{\alpha\in R^+}\frac{(u_\alpha e^{-w\alpha})_\infty}{(e^{-w\alpha})_\infty}\frac{(qe^{-w\alpha})_\infty}{(qu_\alpha e^{-w\alpha})_\infty}=\sum_{w\in W}\prod_{\alpha\in R(w)}u_\alpha \tag{4.20}$$

which follows directly from the original identity (4.19) and the definition of the q–Pochhammer symbol (1.1). Setting $u_\alpha=0$ for all $\alpha\in R^+$ gives a rewriting of the Weyl denominator formula

$$\sum_{w\in W}\prod_{\alpha\in R^+}\frac{(qe^{-w\alpha})_\infty}{(e^{-w\alpha})_\infty}=1 \tag{4.21}$$

Specializing $u_\alpha=1$ for all $\alpha\in R^+$ instead, gives

$$|\,W|=\sum_{w\in W}\prod_{\alpha\in R(w)}u_\alpha \tag{4.22}$$

It also follows from Macdonald's paper [29] that the more general BC_n type identity can be written as

$$\sum_{w\in W}\prod_{\alpha\in R^+}\frac{(u_{2\alpha}^{1/2}u_\alpha e^{-w\alpha})_\infty}{(u_{2\alpha}^{1/2}e^{-w\alpha})_\infty}\frac{(qu_{2\alpha}^{1/2}e^{-w\alpha})_\infty}{(qu_{2\alpha}^{1/2}u_\alpha e^{-w\alpha})_\infty}=\sum_{w\in W}\prod_{\alpha\in R(w)}u_\alpha \tag{4.23}$$

where $u_{2\alpha}^{1/2}=1$ by convention, when 2α is not a root. If one sets $u_{2\alpha}^{1/2}=-1$ for all $\alpha\in R_s^+$, the short roots of R^+, the left hand side of the identity (4.23) becomes

$$\sum_{w\in W}\prod_{\alpha\in R_m^+}\frac{(u_\alpha e^{-w\alpha})_\infty}{(e^{-w\alpha})_\infty}\frac{(qe^{-w\alpha})_\infty}{(qu_\alpha e^{-w\alpha})_\infty}\prod_{\alpha\in R_s^+}\frac{(-u_\alpha e^{-w\alpha})_\infty}{(-e^{-w\alpha})_\infty}\frac{(-qe^{-w\alpha})_\infty}{(-qu_\alpha e^{-w\alpha})_\infty} \tag{4.24}$$

where R_m^+ denotes the medium positive roots.

Some more terminology will be needed in the proof below. Under the action of the Weyl group on R_z, the image of L^+ may not be the full weight lattice L. Depending on z, some subsets of L may be mapped more than once, and some subsets may not be the image of any subset of L^+ under this action.

Let P, called an overlap, denote the (possibly empty) subset of L defined by

$$P=\{\lambda\in L:\lambda\in w_iL^+\cap w_jL^+,\ \text{where } i\neq j \text{ and } i,j\in[2^n n!]\} \tag{4.25}$$

In order to avoid pathological situations with complicated overlap sets, a further restriction is imposed on $z\in\mathbb{R}^n$, and it is required that the $-z$ (the new center) is located outside the fundamental chamber C so that the translated fundamental chamber C_z properly contains C. Under this restriction, the overlap P is non–empty only when $-z$ is on one of the walls of C, and the overlap P turns out to be a proper subset of the walls of the Weyl chambers for the root system R.

Let Q, called a gap, denote the (possibly empty) subset of L containing all $\mu \in L$ that is not an image of any $\lambda \in L^+$ under the standard action of the Weyl group on R_z. In other words, Q is the set theoretical difference

$$Q = L \backslash \cup_{w\in W} wL^+ \tag{4.26}$$

The technique that will be used in multilateralizing basic hypergeometric series or q–series associated to root systems is developed in the next lemma.

Lemma 4.3 (Multilateralization)**.** *Suppose that the summand $f(\zeta+\mu)$ of the sum*

$$\sum_{\mu\in L_+} f(\zeta+\mu) \tag{4.27}$$

has the invariance property that $f(\zeta+\mu) = f(w(\zeta+\mu))$ for any $\zeta \in \mathbb{R}^n$. Fix $z \in \mathbb{R}^n$ such that L is invariant under the action of the Weyl group on R_z as defined by condition (4.16). Choose the fundamental Weyl chamber C, so that $L^+ \subseteq C$ and further require that $C \subseteq C_z$. Another obvious restriction on the choice of z is that $f(z+\mu)$ is well defined (has no essential poles) at any $\mu \in L^+$. Let P denote the overlap and Q denote the gap with respect to this action. Suppose further that

$$\sum_{\mu\in Q} g(z+\mu) = 0 \tag{4.28}$$

and

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in P\cap L^+} f(z+\mu) = \sum_{\mu\in P} g(z+\mu) \tag{4.29}$$

where g is defined by

$$g(z+\mu) := f(z+\mu)\left(\prod_{\alpha\in R^+}\frac{(u_{2\alpha}^{1/2}u_\alpha q^{\langle\alpha,\, z+\mu\rangle})_\infty}{(u_{2\alpha}^{1/2}q^{\langle\alpha,\, z+\mu\rangle})_\infty}\frac{(qu_{2\alpha}^{1/2}q^{\langle\alpha,\, z+\mu\rangle})_\infty}{(qu_{2\alpha}^{1/2}u_\alpha q^{\langle\alpha,\, z+\mu\rangle})_\infty}\right) \tag{4.30}$$

in terms of the indeterminates u_α. Under these conditions,

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in L^+} f(z+\mu) = \sum_{\mu\in L} g(z+\mu) \tag{4.31}$$

Proof. The identity (4.23) can be written in the form

$$\sum_{w\in W}\prod_{\alpha\in R^+}\frac{(u_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}{(u_{2\alpha}^{1/2}q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}\frac{(qu_{2\alpha}^{1/2}q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}{(qu_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty} = \sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha \tag{4.32}$$

Next, write

$$\sum_{\mu\in L^+} f(z+\mu) = \sum_{\mu\in P\cap L^+} f(z+\mu) + \sum_{\mu\in L^+\backslash P} f(z+\mu) \tag{4.33}$$

and consider the second sum. Multiplying $\sum_{\mu\in L^+\backslash P} f(z+\mu)$ by the right hand side of the above identity (4.32) one gets

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in L^+\backslash P} f(z+\mu) \tag{4.34}$$

$$= \sum_{\mu\in L^+\backslash P}\sum_{w\in W} f(w(z+\mu))\prod_{\alpha\in R^+}\frac{(u_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}{(u_{2\alpha}^{1/2}q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}\frac{(qu_{2\alpha}^{1/2}q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}{(qu_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, w(z+\mu)\rangle})_\infty}$$

Note that $f(z+\mu)$ is replaced by $f(w(z+\mu))$ on the right hand side due to the W invariance of the summand f. If the order of summation is switched and the sum is written as $\sum_{w\in W}\sum_{\mu\in L^+\backslash P}$, it becomes clear from the definitions that this double sum can be written as a single sum over the subset $L\backslash(Q\cup P)$. Because, this is precisely the set that contains all $\lambda\in L$ such that $w(\mu+z)=\lambda+z$ for some $\mu\in L^+\backslash P$ and $w\in W$. Thus, it follows that

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in L^+\backslash P} f(z+\mu) \tag{4.35}$$

$$= \sum_{\mu\in L\backslash(Q\cup P)} f(z+\mu)\left(\prod_{\alpha\in R^+}\frac{(u_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, z+\mu\rangle})_\infty}{(u_{2\alpha}^{1/2}q^{-\langle\alpha,\, z+\mu\rangle})_\infty}\frac{(qu_{2\alpha}^{1/2}q^{-\langle\alpha,\, z+\mu\rangle})_\infty}{(qu_{2\alpha}^{1/2}u_\alpha q^{-\langle\alpha,\, z+\mu\rangle})_\infty}\right)$$

But due to the condition (4.28) this sum on the right hand side can actually be written over $L\backslash P$. Therefore, assuming that the condition (4.28) holds, one gets

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in L^+\backslash P} f(z+\mu) = \sum_{\mu\in L\backslash P} g(z+\mu) \tag{4.36}$$

Finally adding the sum $\sum_{\mu\in P} g(z+\mu)$ to both sides and using the condition (4.29) gives

$$\left(\sum_{w\in W}\prod_{\alpha\in R(w)} u_\alpha\right)\sum_{\mu\in L^+} f(z+\mu) = \sum_{\mu\in L} g(z+\mu) \tag{4.37}$$

as desired. □

Note that the argument used in the proof of Lemma 4.3 generalizes the technique used in Section 2 for the alternative proof of a generalization of the Rogers–Ramanujan identities.

In order to show that the multiple Rogers–Selberg identity (4.1) can be multilateralized, the symmetries of the summands in both sides should be verified.

Lemma 4.4. *The summand of the series in the well–poised side of the Rogers–Selberg identity (4.1) has a C_n type symmetry, and the one in the balanced side has an A_n type symmetry.*

Proof. That the series in the well–poised side of (4.1) has C_n type symmetries is already established in a more general setting in Remark 3.20.

The series in the balanced side is invariant under the standard action of the Weyl group of the root system A_n of rank n (i.e., the symmetric group in n letters). This can be seen similarly by setting $q^{z_i} = b^{1/2}t^{1-i}$ and again using (1.1), (3.44) and

$$\prod_{1\le i<j\le n} a_i = \prod_{i=1}^{n} a_i^{n-i} \tag{4.38}$$

to rewrite the series in the following form.

$$\begin{aligned}\prod_{i=1}^{n} q^{-z_i^2}\frac{(q^{1+2z_i}t^{i-1})_\infty}{(qt^{n-i})_\infty} &\prod_{1\le i<j\le n}\left\{\frac{(q^{1+z_i-z_j})_\infty}{(t^{-1}q^{1+z_i-z_j})_\infty}\frac{(q^{1+z_j-z_i})_\infty}{(t^{-1}q^{1+z_j-z_i})_\infty}\right\}\\ &\cdot\sum_{\mu,\ell(\mu)\le n}\prod_{i=1}^{n}\left\{q^{(z_i+\mu_i)^2}(b^{-1/2}t^{n-1}q^{1+\mu_i+z_i})_\infty\right\}\\ &\cdot\prod_{1\le i<j\le n}\left\{\frac{(t^{-1}q^{1+z_i+\mu_i-z_j-\mu_j})_\infty}{(q^{1+z_i+\mu_i-z_j-\mu_j})_\infty}\frac{(t^{-1}q^{1-z_i-\mu_i+z_j+\mu_j})_\infty}{(q^{1-z_i-\mu_i+z_j+\mu_j})_\infty}\right\}\end{aligned} \tag{4.39}$$

It is obvious from this representation that the series on the balanced side is invariant under the permutations $q^{z+\mu} \leftrightarrow q^{w(z+\mu)}$ of the symmetric group. □

Remark 4.5. The crucial point is to fix a center $-z \in \mathbb{Z}^n$ for the multilateralization lemma satisfying all the restrictions imposed by Lemma 4.3. The condition (4.16) implies $w(z) - z \in L$, which in turn implies that each z_i is of the form $m_i/2$ for integers m_i and that all m_i for $i \in [n]$ have the same parity. This observation together with the hypothesis $C \subseteq C_z$ in Lemma 4.3 implies that $z = m/2 + \mu = (m/2+\mu_1, \ldots, m/2+\mu_n)$ where m is a non–negative integer and μ is some partition with at most n parts and $\mu_n = 0$. On the other hand, the substitution $q^{z_i} = b^{1/2}t^{1-i}$ made in the series in (4.1) yields that $z_i = 1/2\log_q b + (1-i)\log_q t$.

Combine the two equivalent representations $z_i = 1/2\log_q b + (1-i)\log_q t$ and $z_i = m/2 + \mu_i$ from the last paragraph to show that multilateralization is possible when $t = q^k$ for some non–negative integer $k = \mu_{n-1}$. It also turns out that $b = q^{m+2(n-1)k}$ and, in general, z has to have the form

$$z_i = m/2 + k(n-i) \tag{4.40}$$

where m and k are non–negative integers.

It is worthwhile to note that the assumption $C \subseteq C_z$ is not essential in the proof of Lemma 4.3 above. It guarentees that the overlap and gap sets are not too complicated in applications. In principle, this assumption may be dropped leading, in particular, to specializations where m and k could be negative. In fact, an example is presented in Section 2 where m was an arbitrary integer.

The following notation will be used in the sequel. $\aleph_\lambda$ is defined by

$$\aleph_\lambda = \begin{cases} 1, \text{ if } \lambda \text{ is a rectangular partition} \\ n!, \text{otherwise}\end{cases} \tag{4.41}$$

and the product $\prod^{r,s}$ denotes

$$\prod\nolimits^{r,s} f(x) = \begin{cases} s, \text{ if } r = 0 \\ \prod f(x), \text{otherwise}\end{cases} \tag{4.42}$$

Lemma 4.6 (B_n and D_n Specializations)**.** *Set $q^{z_i} = b^{1/2}t^{1-i}$ for $i \in [n]$ in the unspecialized Rogers–Selberg identity (4.1). With the specialization $z_i = m/2 + k(n-i)$, where m and k are non–negative integers, the series on the right hand side of the identity (4.1) can be written as follows.*

$$\begin{aligned}(4.43)\quad &\prod_{i=1}^{n}(q^{1+k(n-i)})_\infty \cdot \sum_{\lambda,\ell(\lambda)<n}\prod_{i=1}^{n}\left\{q^{(m+k(n-1))\lambda_i+\lambda_i^2}\frac{1}{(q^{1+k(n-i)})_{\lambda_i}}\right\}\\ &\quad\cdot\prod_{1\le i<j\le n}^{k,\aleph_\lambda}\left\{\frac{(q^{1+k(-1-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{1+k(-i+j)+\lambda_i-\lambda_j})_\infty}\frac{(q^{k(-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{k(1-i+j)+\lambda_i-\lambda_j})_\infty}\right\}\\ &=\sum_{\lambda\in\mathbb{Z}^n}\prod_{i=1}^{n}\left\{(-1)^{\lambda_i}\,q^{(-1/2+2m+3k(n-i))\lambda_i+5\lambda_i^2/2}\right\}\prod_{i=1}^{n}{}^{m,1}\left\{\frac{(q^{1+k(n-i)+\lambda_i})_\infty}{(q^{m+k(n-i)+\lambda_i})_\infty}\right\}\\ &\quad\cdot\prod_{1\le i<j\le n}^{k,1}\left\{\frac{(q^{1+k(-1-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{m+k(1+2n-i-j)+\lambda_i+\lambda_j})_\infty}\frac{(q^{1+m+k(-1+2n-i-j)+\lambda_i+\lambda_j})_\infty}{(q^{k(1-i+j)+\lambda_i-\lambda_j})_\infty}\right\}\end{aligned}$$

The $m > 1$ case corresponds to the B_n specializations and that of $m \in \{0,1\}$ corresponds to the D_n specializations.

Proof. It follows from (4.5) and (4.6) that the balanced side and the well–poised side of the Rogers–Selberg identity (4.1) can be written in the form

$$\begin{aligned}(4.44)\quad &\prod_{i=1}^{n}(q^{1+2z_i}t^{i-1})_\infty\cdot\sum_{\lambda,\ell(\lambda)\le n}\prod_{i=1}q^{2z_i\lambda_i+\lambda_i^2}t^{(-1-n+2i)\lambda_i}\frac{1}{(qt^{n-i})_{\lambda_i}}\\ &\quad\cdot\prod_{1\le i<j\le n}\left\{\frac{(qt^{-1}q^{z_i-z_j+\lambda_i-\lambda_j})_\infty}{(q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty}\frac{(q^{z_i-z_j+\lambda_i-\lambda_j})_\infty}{(tq^{z_i-z_j+\lambda_i-\lambda_j})_\infty}\right\}\\ &=\prod_{i=1}^{n}\frac{(q^{2z_i}t^{i-n})_\infty}{(qt^{n-i})_\infty}\frac{(q^{1+2z_i})_\infty}{(q^{2z_i})_\infty}\cdot\prod_{1\le i<j\le n}\left\{\frac{(tq^{z_i+z_j})_\infty}{(q^{z_i+z_j})_\infty}\frac{(q^{1+z_i+z_j})_\infty}{(qt^{-1}q^{z_i+z_j})_\infty}\right\}\\ &\cdot\sum_{\ell(\lambda)\le n}\prod_{i=1}^{n}(-1)^{\lambda_i}\,q^{4z_i\lambda_i}t^{(i-n)\lambda_i}q^{5\lambda_i^2/2-\lambda_i/2}\frac{(t^{n-i}q^{1+\lambda_i})_\infty}{(q^{2z_i}t^{i-n}q^{\lambda_i})_\infty}\frac{(q^{2z_i+2\lambda_i})_\infty}{(q^{1+2z_i+2\lambda_i})_\infty}\\ &\quad\cdot\prod_{1\le i<j\le n}\left\{\frac{(t^{-1}q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty}{(q^{1+z_i-z_j+\lambda_i-\lambda_j})_\infty}\frac{(q^{z_i+z_j+\lambda_i+\lambda_j})_\infty}{(tq^{z_i+z_j+\lambda_i+\lambda_j})_\infty}\right.\\ &\qquad\qquad\left.\cdot\frac{(q^{z_i-z_j+\lambda_i-\lambda_j})_\infty(t^{-1}q^{1+z_i+z_j+\lambda_i+\lambda_j})_\infty}{(tq^{z_i-z_j+\lambda_i-\lambda_j})_\infty(q^{1+z_i+z_j+\lambda_i+\lambda_j})_\infty}\right\}\end{aligned}$$

The series on the right hand side of (4.44) will be multilateralized. Remark 4.5 preceding this lemma explain, in the notation of Lemma 4.3, that $L = \mathbb{Z}^n$ is invariant under the standard action of the hyperoctahedral group W (the semidirect product of the symmetric group $\mathbb{S}_n$ and $\mathbb{Z}_2^n$) on R_z. That is, under the maps $q^{z_i+\mu_i} \leftrightarrow q^{\pm(z_j+\mu_j)}$ for all $i, j \in [n]$, the lattice $L = \mathbb{Z}^n$ remains invariant when $z_i = m/2 + k(n-i)$ and $t = q^k$.

First note that the front factors on both sides, under this specialization, combine to give

$$\prod_{i=1}^{n}\frac{1}{(q^{1+k(n-i)})_\infty}=\prod_{1\le i<j\le n}\left\{\frac{(q^{m+k(1+2n-i-j)})_\infty}{(q^{m+k(2n-i-j)})_\infty}\frac{(q^{1+m+k(2n-i-j)})_\infty}{(q^{1+m+k(-1+2n-i-j)})_\infty}\right\}\cdot\prod_{i=1}^{n}\left\{\frac{(q^{m+k(n-i)})_\infty}{(q^{1+k(n-i)})_\infty}\frac{(q^{1+m+2k(n-i)})_\infty}{(q^{m+2k(n-i)})_\infty}\frac{1}{(q^{1+m+k(2n-i-1)})_\infty}\right\}\tag{4.45}$$

for any positive integer m.

Since the specializations with either $m=0$ or $k=0$ produce non–empty overlap sets, the proof will be divided into four cases depending on whether m or k is zero. For each case the subsets P and Q of L and corresponding Macdonald polynomial identity will be identified, and the conditions of Lemma 4.3 will be verified.

Two types of subsets of the weight lattice $L=\mathbb{Z}^n$ are special in the application of Lemma 4.3. They are defined as the intersection of L and the following two finite affine hyperplane arrangements:

(1) $x_i=-m-k(n-i)+1+\ell$ where $i\in[n]$, and $\ell\ge -1$ is an integer.
(2) (a) $x_i=x_j-k(j-i+1)+1+\ell$, and
 (b) $x_i=-x_j-m-k(2n-j-i+1)+1+\ell$
 where $1\le i<j\le n$, and $\ell\ge -1$ is an integer.

Case 1: $m,k\neq 0$. In this case the gap Q consists of the Type 1 subsets for all $\ell=0,1,\dots,m-2$ and the Type 2 subsets for all $\ell=0,1,\dots,2(k-1)$. The overlap P is empty, that is $P=\emptyset$.

Case 2: $m=0,k\neq 0$. Here, the gap Q consists of the Type 2 subsets for all $\ell=0,1,\dots,2(k-1)$. The overlap P is the set theoretical difference of the Type 1 subsets with $\ell=-1$ and the gap Q.

Case 3: $m\neq 0,k=0$. Similarly, in this case the gap Q consists of the Type 1 subsets for all $\ell=0,1,\dots,m-2$. The overlap P is the set theoretical difference of the Type 2 subsets with $\ell=-1$ and the gap Q.

Case 4: $m=0,k=0$. Finally, the gap Q is empty if both m and k are zeros. The overlap P consists of the Type 1 and Type 2 subsets with $\ell=-1$.

For the cases where $m\neq 0$, the C_n type Macdonald's polynomial identity (4.20) will be used with $u_\alpha=0$ for all positive roots α. This identity takes on the form

$$\sum_{w\in W}\prod_{1\le i<j\le n}\left\{\frac{(qw(x_i^{-1}x_j))_\infty}{(w(x_i^{-1}x_j))_\infty}\frac{(qw(x_i^{-1}x_j^{-1}))_\infty}{(w(x_i^{-1}x_j^{-1}))_\infty}\right\}\prod_{i=1}^{n}\frac{(qw(x_i^{-2}))_\infty}{(w(x_i^{-2}))_\infty}=1\tag{4.46}$$

where $x_i=q^{z_i+\lambda_i}=q^{m/2+k(n-i)+\lambda_i}$ and W is the hyperoctahedral group. Setting $u_\alpha=0$ will remove a set of factors corresponding to the positive roots for C_n and therefore yields the B_n type specialization.

When $m=0$, however, the BC_n type identity (4.24) will be used with $u_\alpha=0$ for all roots of medium length. The identity can be written as

$$\sum_{w\in W}\prod_{1\le i<j\le n}\left\{\frac{(qw(x_i^{-1}x_j))_\infty}{(w(x_i^{-1}x_j))_\infty}\frac{(qw(x_i^{-1}x_j^{-1}))_\infty}{(w(x_i^{-1}x_j^{-1}))_\infty}\right\}\prod_{i=1}^{n}\frac{(-qw(x_i^{-1}))_\infty}{(-w(x_i^{-1}))_\infty}=1\tag{4.47}$$

where x_i and W are defined as before. This choice will remove a set of factors corresponding to the long, the short and the medium positive roots for BC_n and therefore yields the D_n type specialization.

In both cases, one gets

$$g(z+\lambda)=\prod_{1\le i<j\le n}^{k,1}\left\{\frac{(q^{1+k(-1-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{m+k(1+2n-i-j)+\lambda_i+\lambda_j})_\infty}\frac{(q^{1+m+k(-1+2n-i-j)+\lambda_i+\lambda_j})_\infty}{(q^{k(1-i+j)+\lambda_i-\lambda_j})_\infty}\right\}$$
$$\cdot\prod_{i=1}^{n}\left\{(-1)^{\lambda_i}\,q^{(-1/2+2m+3k(n-i))\lambda_i+5\lambda_i^2/2}\right\}\prod_{i=1}^{n}{}^{m,1}\left\{\frac{(q^{1+k(n-i)+\lambda_i})_\infty}{(q^{m+k(n-i)+\lambda_i})_\infty}\right\} \tag{4.48}$$

where the summand $g(z+\lambda)$ is as defined in Lemma 4.3.

The rest of the proof shows that the gap condition (4.28) and the overlap condition (4.29) in Lemma 4.3 are both satisfied. The factors of the form

$$\prod_{i=1}^{n}{}^{m,1}\left\{\frac{(q^{1+k(n-i)+\lambda_i})_\infty}{(q^{m+k(n-i)+\lambda_i})_\infty}\right\},\ \prod_{1\le i<j\le n}^{k,1}\left\{\frac{(q^{1+m+k(-1+2n-i-j)+\lambda_i+\lambda_j})_\infty}{(q^{m+k(1+2n-i-j)+\lambda_i+\lambda_j})_\infty}\right\} \tag{4.49}$$

and

$$\prod_{1\le i<j\le n}^{k,1}\left\{\frac{(q^{1+k(-1-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{k(1-i+j)+\lambda_i-\lambda_j})_\infty}\right\} \tag{4.50}$$

in (4.44) show that the summand $g(z+\lambda)$ vanishes over Type 1, Type 2 (a) and Type 2 (b) subsets, respectively, verifying the gap condition (4.28) in all four cases where the gap Q is non–empty.

In all three cases where the overlap P is non–empty, the series over $P\cap L^+$ is removed from the series over L^+, and the resulting series

$$\sum_{\mu\in L^+\setminus(P\cap L^+)} f(z+\mu)\quad\text{and}\quad\sum_{\mu\in P\cap L^+} f(z+\mu) \tag{4.51}$$

are multilateralized separately using Lemma 4.3, giving

$$\sum_{\mu\in L^+\setminus(P\cap L^+)} f(z+\mu)=\sum_{\mu\in L\setminus P} g(z+\mu) \tag{4.52}$$

and

$$\sum_{\mu\in P\cap L^+} f(z+\mu)=\sum_{\mu\in P} g(z+\mu) \tag{4.53}$$

For the second series over $P\cap L^+$ in (4.51), a special version of Lemma 4.3 is needed. Namely, L is replaced by P and L^+ is replaced by $P\cap L^+$ in the lemma, and appropriate versions of Macdonald's identities (4.46) and (4.47) are used depending on whether $m=0$ or $m>0$ by specializing u_α parameters accordingly. Finally, the two multilateral series are added together to get the single multilateral series $\sum_{\mu\in L} g(z+\mu)$. □

Remark 4.7. The main argument in the proof of of Lemma 4.3 can be applied to the series on the balanced side of Rogers–Selberg identity (4.1) to write it as a sum over $\mathbb{Z}^n_{\geq}$. Here, one uses the following special case of the A_n version of Macdonald's polynomial identity (4.20)

$$\sum_{w\in W}\prod_{1\le i<j\le n}\frac{(qw(x_i^{-1}x_j))_\infty}{(w(x_i^{-1}x_j^{-1}))_\infty}=1 \tag{4.54}$$

where $W = S_n$, the symmetric group on n letters. This is because the series on the balanced side has A_n type symmetries. Multiplying both sides by $\prod_{i=1}^{n}(q)_{k(n-i)}$ using

$$(q)_{\lambda_i} = (q)_{k(n-i)}(q^{1+k(n-i)})_{\lambda_i-k(n-i)} \tag{4.55}$$

the specialized BC_n Rogers–Selberg identity, therefore, can also be written as

$$\sum_{\lambda\in\mathbb{Z}^n_{\geq}}\prod_{i=1}^{n}\left\{\frac{q^{(m+k(n-1))(\lambda_i-k(n-i))+(\lambda_i-k(n-i))^2}}{(q)_{\lambda_i}}\right\}\prod_{1\leq i<j\leq n}^{k,1}\left\{\frac{(q^{1-k+\lambda_i-\lambda_j})_\infty}{(q^{k+\lambda_i-\lambda_j})_\infty}\right\} \tag{4.56}$$

$$= \frac{1}{(q)^n_\infty}\cdot\sum_{\lambda\in\mathbb{Z}^n}\prod_{i=1}^{n}\left\{(-1)^{\lambda_i}\, q^{(-1/2+2m+3k(n-i))\lambda_i+5\lambda_i^2/2}\right\}\prod_{i=1}^{n}{}^{m,1}\left\{\frac{(q^{1+k(n-i)+\lambda_i})_\infty}{(q^{m+k(n-i)+\lambda_i})_\infty}\right\}$$

$$\cdot\prod_{1\leq i<j\leq n}^{k,1}\left\{\frac{(q^{1+k(-1-i+j)+\lambda_i-\lambda_j})_\infty}{(q^{m+k(1+2n-i-j)+\lambda_i+\lambda_j})_\infty}\frac{(q^{1+m+k(-1+2n-i-j)+\lambda_i+\lambda_j})_\infty}{(q^{k(1-i+j)+\lambda_i-\lambda_j})_\infty}\right\}$$

4.3. **Multiple Rogers–Ramanujan Identities.** Recall from Section 2 that the two bilateralizations obtained for the one dimensional Rogers–Selberg identity yielded the two Rogers–Ramanujan identities (1.2) by an application of the Jacobi triple product identity (2.7). A well–known generalization of the Jacobi triple product identity to arbitrary root systems is called Macdonald identities [28], [23]. One expects that a similar procedure would produce product representations for the specialized B_n and D_n Rogers–Selberg identities (4.43) or (4.56) using the more general Macdonald identities. This proves to be a non–trivial problem except in a special case when $k = 0$. It is possible, however, to give interesting generalizations of the classical Rogers–Ramanujan identities in terms of determinants of theta functions for the root systems B_n and D_n of rank n, at least for the case $k = 1$.

First, it will be noted that the trivial specializations corresponding to $k = 0$ (i.e., $t = 1$) result in product representations.

Theorem 4.8. *The D_n specializations (4.43) of the Rogers–Selberg identity give n-fold products of the first and second Rogers–Ramanujan identities (1.2), respectively. Namely,*

$$\sum_{\lambda\in\mathbb{Z}^n_{\geq}}\prod_{i=1}^{n}\left\{\frac{q^{\delta\lambda_i+\lambda_i^2}}{(q)_{\lambda_i}}\right\} = \prod_{i=1}^{n}\left\{\frac{1}{(q^{1+\delta};q^5)_\infty(q^{4-\delta};q^5)_\infty}\right\} \tag{4.57}$$

where $\delta\in\{0,1\}$ and $|q|<1$ as usual.

Proof. The proof follows, immediately, from the $k = 0$ case of (4.43) and (4.56) which can be written as

$$\sum_{\lambda,\,\ell(\lambda)\leq n}\aleph_\lambda\cdot\prod_{i=1}^{n}\left\{q^{m\lambda_i+\lambda_i^2}\frac{1}{(q)_{\lambda_i}}\right\} = \sum_{\lambda\in\mathbb{Z}^n_{\geq}}\prod_{i=1}^{n}\left\{\frac{q^{m\lambda_i+\lambda_i^2}}{(q)_{\lambda_i}}\right\} \tag{4.58}$$

$$= \prod_{i=1}^{n}\left\{\frac{1}{(q)_\infty}\right\}\cdot\sum_{\lambda\in\mathbb{Z}^n}\prod_{i=1}^{n}\left\{(-1)^{\lambda_i}\, q^{(-1/2+2m)\lambda_i+5\lambda_i^2/2}\prod_{r=1}^{m-1}(1-q^{r+\lambda_i})\right\}$$

where the product $\prod_{r=1}^{m-1}a_r = 1$ for $m\leq 1$. Setting $m=\delta=0$ and $m=\delta=1$ and applying the one dimensional Jacobi triple product identity (2.7) gives the result to be proved. □

Among all possible specializations $z_i = m/2 + k(n-i)$ in (4.40), the case $k=1$ gives similar simplifications to the one dimensional case when $m=0$ and $m=1$. Namely, the factors corresponding to the short and long roots in the BC_n Rogers–Selberg identity (4.1) simplify in these cases.

Theorem 4.9 (B_n and D_n Rogers–Ramanujan identities)**.** *With the notation as above, D_n multiple Rogers–Ramanujan identities can be written as*

$$\sum_{\lambda\in\mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(\delta+n-1)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \{(1-q^{\lambda_i-\lambda_j})\} \tag{4.59}$$
$$= \frac{1}{2}(-1)^{\binom{n}{2}} \prod_{i=1}^{n} \left\{ \frac{(q^5;q^5)_\infty}{(q)_\infty} q^{(n-i)(n-i+\delta/2)} \right\}$$
$$\cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1)(n-i+\delta/2)}\, \theta(q^{4n+2\delta+1-4i+j};q^5) + q^{-(j-1)(n-i+\delta/2)}\, \theta(q^{4n+2\delta+3-4i-j};q^5) \Big)$$

The cases $m=\delta=0$ and $m=\delta=1$ give the first and the second D_n Rogers–Ramanujan identities, respectively.

A single B_n multiple Rogers–Ramanujan identity which corresponds to $m=2$ may be written, similarly, in the form

$$\sum_{\lambda\in\mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(1+n)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \{(1-q^{\lambda_i-\lambda_j})\} \tag{4.60}$$
$$= (-1)^{\binom{n}{2}+n} \prod_{i=1}^{n} \left\{ \frac{(q^5;q^5)_\infty}{(q)_\infty} q^{(n-i+1/2)(n-i+1)} \right\}$$
$$\cdot \det_{1\leq i,j\leq n} \Big(q^{(j-1/2)(n-i+1)} \theta(q^{6+4n-4i+j};q^5) - q^{-(j-1/2)(n-i+1)} \theta(q^{7+4n-4i-j};q^5) \Big)$$

In both cases, n is a positive integer and $|q|<1$ as usual.

Proof. The $k=1$ case of (4.56) reduces to

$$\sum_{\lambda\in\mathbb{Z}^n_{\geq}} \prod_{i=1}^{n} \left\{ \frac{q^{(m+n-1)(\lambda_i-n+i)+(\lambda_i-n+i)^2}}{(q)_{\lambda_i}} \right\} \prod_{1\leq i<j\leq n} \{(1-q^{\lambda_i-\lambda_j})\} \tag{4.61}$$
$$= \prod_{i=1}^{n} \frac{1}{(q)_\infty} \sum_{\lambda\in\mathbb{Z}^n} \prod_{i=1}^{n} \left\{ (-1)^{\lambda_i} q^{(-1/2+2m+3(n-i))\lambda_i+5\lambda_i^2/2} \prod_{r=1}^{m-1} (1-q^{r+n-i+\lambda_i}) \right\}$$
$$\cdot \prod_{1\leq i<j\leq n} \{(1-q^{-i+j+\lambda_i-\lambda_j})(1-q^{m+2n-i-j+\lambda_i+\lambda_j})\}$$

where the product $\prod_{r=1}^{m-1} a_r = 1$ for $m \leq 1$.

Setting $m = \delta \in \{0,1\}$ and using the well–known determinant evaluations [26]

$$\prod_{1\le i<j\le n} (1 - x_i x_j^{-1})(1 - x_i x_j) = \frac{1}{2}(-1)^{\binom{n}{2}} \prod_{i=1}^{n} x_i^{n-i} \det_{1\le i,j\le n}\left(x_i^{j-1} + x_i^{-(j-1)}\right) \tag{4.62}$$

the series on the well–poised side of (4.61) becomes

$$\frac{1}{2}(-1)^{\binom{n}{2}} \prod_{i=1}^{n}\left\{q^{(n-i)(n-i+\delta/2)}\right\} \det_{1\le i,j\le n}\left(\sum_{\lambda_i\in\mathbb{Z}} (-1)^{\lambda_i} q^{2(1+\delta+2n-2i)\lambda_i + 5\binom{\lambda_i}{2}} \cdot \left(q^{(j-1)(\lambda_i+n-i+\delta/2)} + q^{-(j-1)(\lambda_i+n-i+\delta/2)}\right)\right) \tag{4.63}$$

The one dimensional Jacobi triple product identity (2.7) is then used to get the desired D_n identities.

Setting $m = 2$ in (4.61), and pursuing a similar line of thought using another determinant evaluation [26]

$$\prod_{1\le i<j\le n} (1 - x_i x_j^{-1})(1 - x_i x_j) \prod_{i=1}^{n}(1 - x_i) = (-1)^{\binom{n}{2}+n} \prod_{i=1}^{n} x_i^{n-i+1/2} \det_{1\le i,j\le n}\left(x_i^{j-1/2} - x_i^{-(j-1/2)}\right) \tag{4.64}$$

gives the desired B_n Rogers–Ramanujan identity. □

Remark 4.10. The Vandermonde determinant

$$\prod_{1\le i<j\le n} (1 - x_i x_j^{-1}) = (-1)^{\binom{n}{2}} \prod_{i=1}^{n} x_i^{1-i} \det_{1\le i,j\le n}\left(x_i^{n-j}\right) \tag{4.65}$$

may be used similarly to write the balanced sides of (4.61) as a determinant of theta functions as well, which produces a determinant transformation identity where an A_n type determinant equals a B_n or D_n type determinant of theta functions.

Using (4.65) and the recent result (2.1), the determinant transformation identities corresponding to (4.59) and (4.60) can be written in the form

$$2q^{\binom{n}{2}(1-n-3\delta/2)} \det_{1\le i,j\le n}\left(\pi_{\delta+i-j}(q)\right) = \det_{1\le i,j\le n}\left(q^{(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+1-4i+j};q^5) + q^{-(j-1)(n-i+\delta/2)}\,\theta(q^{4n+2\delta+3-4i-j};q^5)\right) \tag{4.66}$$

and

$$(4.67)\quad (-1)^{\binom{n}{2}} q^{-n(2n^2+3n-3)/4} \det_{1\le i,j\le n}\left(\pi_{2+i-j}(q)\right) = \det_{1\le i,j\le n}\Big(q^{(j-1/2)(n-i+1)}\theta(q^{6+4n-4i+j};q^5) - q^{-(j-1/2)(n-i+1)}\theta(q^{7+4n-4i-j};q^5)\Big)$$

respectively, where

$$(4.68)\quad \pi_k(q) := (-1)^k q^{-\binom{k}{2}}\theta(q^2;q^5)E_{k-2}(q) - (-1)^k q^{-\binom{k}{2}}\theta(q;q^5)D_{k-2}(q)$$

for $k\in\mathbb{Z}$, and the Schur polynomials $D_k(q)$ and $E_k(q)$ are defined as in (2.2).

These results do not yield new relations between theta functions, yet they appear to be new determinant transformation identities. A direct proof using manipulations of theta functions and properties of determinants is not obvious beyond dimension $n=2$. This is because the entries of the Toeplitz matrix on the left hand side in all three cases involve Schur polynomials of higher degrees as n gets larger. It should be also noted that the identities can be put into different forms by transposing matrices, etc.

Proof. Using (4.65), the right hand side of the $k=1$ case of (4.56) can be written in the form

$$(4.69)\quad (-1)^{\binom{n}{2}} \prod_{i=1}^{n}\left\{q^{(m-1+i)(-n+i)}\right\} \det_{1\le i,j\le n}\left(\sum_{\lambda_i\in\mathbb{Z}_{\ge}} \frac{q^{(m+i-j)\lambda_i+\lambda_i^2}}{(q)_{\lambda_i}}\right)$$

The identity (2.1) together with the definition (4.68) now imply that the right hand side equals

$$(4.70)\quad (-1)^{\binom{n}{2}} \prod_{i=1}^{n}\left\{q^{(m-1+i)(-n+i)}\right\} \det_{1\le i,j\le n}\left(\pi_{m+i-j}(q)\right)$$

The determinant transformation identities to be proved now follow. □

Multiple generalizations of other important one dimensional q–series identities can proved using the BC_n Bailey Lemma 3.17 and the Lemma 4.3. In certain cases, it is possible to write such multiple generalizations as determinant transformation identities involving theta functions similar to the ones given above in (4.66) and (4.67). In other cases, it is possible to find non–trivial product representations producing genuine extensions of the classical results. The next section discusses a remarkable such generalization of Euler's Pentagonal Number Theorem.

4.4. **Euler's Pentagonal Number Theorem.** An infinite family of D_n Euler's Pentagonal Number Theorems can be written as follows.

Lemma 4.11. *Let n be a positive and k be a non–negative integer. Then*

$$(4.71)\quad (q)_\infty^n \prod_{1\le i<j\le n}\frac{(q^{k(j-i)})_\infty}{(q^{k(j-i+1)})_\infty} = \sum_{\mu\in\mathbb{Z}^n}(-1)^{|\mu|}q^{-kn(\mu)+3n(\mu')+|\mu|(k(n-1)+1)} \cdot \prod_{1\le i<j\le n}^{k,1}\left\{\frac{(q^{1+k(j-i-1)+\mu_i-\mu_j})_\infty}{(q^{k(1+2n-i-j)+\mu_i+\mu_j})_\infty}\frac{(q^{1+k(-1+2n-i-j)+\mu_i+\mu_j})_\infty}{(q^{k(j-i+1)+\mu_i-\mu_j})_\infty}\right\}$$

where $q \in \mathbb{C}$ such that $|q| < 1$ as usual.

Proof. For a rectangular partition $\lambda = k^n$, the terminating ${}_6\varphi_5$ summation (3.61) can be written in the form

$$(4.72)\quad \frac{(qb, qb/\rho_1\sigma_1)_{k^n}}{(qb/\sigma_1, qb/\rho_1)_{k^n}} = \sum_{\mu\subseteq\lambda} t^{2n(\mu)+(1-n)|\mu|}\left(\frac{q^{1+k}b}{\sigma_1\rho_1}\right)^{|\mu|} \cdot \prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}\frac{(bt^{1-n},\sigma_1,\rho_1,q^{-k})_\mu}{(qt^{n-1},qb/\sigma_1,qb/\rho_1,q^{1+k}b)_\mu} \cdot \prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\frac{(t^{j-i+1})_{\mu_i-\mu_j}(qbt^{2-i-j})_{\mu_i+\mu_j}}{(t^{j-i})_{\mu_i-\mu_j}(qbt^{1-i-j})_{\mu_i+\mu_j}}\right\}$$

by the virtue of the Weyl degree formula (3.41). Sending ρ_1, σ_1 and k to ∞ using the Dominated Convergence Theorem, similar to the proof of the Rogers–Selberg identity in Lemma 4.1, gives

$$(4.73)\quad (qb)_{\infty^n} = \sum_{\mu,\ell(\mu)\le n} t^{-n(\mu)+(1-n)|\mu|}b^{|\mu|}(-1)^{|\mu|}q^{|\mu|+3n(\mu')} \cdot \prod_{i=1}^{n}\left\{\frac{(1-bt^{2-2i}q^{2\mu_i})}{(1-bt^{2-2i})}\right\}\frac{(bt^{1-n})_\mu}{(qt^{n-1})_\mu} \cdot \prod_{1\le i<j\le n}\left\{\frac{(qt^{j-i})_{\mu_i-\mu_j}(bt^{3-i-j})_{\mu_i+\mu_j}}{(qt^{j-i-1})_{\mu_i-\mu_j}(bt^{2-i-j})_{\mu_i+\mu_j}}\frac{(t^{j-i+1})_{\mu_i-\mu_j}(qbt^{2-i-j})_{\mu_i+\mu_j}}{(t^{j-i})_{\mu_i-\mu_j}(qbt^{1-i-j})_{\mu_i+\mu_j}}\right\}$$

Now let $t = q^k$ and $b = q^{m+2k(n-1)}$. This corresponds to setting $q^{z_i} = b^{1/2}t^{1-i}$ for $i \in [n]$ and specializing $z_i = m/2 + k(n-i)$ for non–negative integers m and k as in (4.40). Flip appropriate terms using the definition of the q–Pochhammer symbol (1.1) and simplify to get

$$(4.74)\quad (q)_\infty^n \prod_{1\le i<j\le n}\frac{(q^{k(j-i)})_\infty}{(q^{k(j-i+1)})_\infty} = \sum_{\mu,\ell(\mu)\le n}(-1)^{|\mu|}q^{-kn(\mu)+3n(\mu')+|\mu|(m+k(n-1)+1)} \cdot \prod_{i=1}^{n}\left\{\frac{(q^{m+2k(n-i)+2\mu_i})_\infty}{(q^{1+m+2k(n-i)+2\mu_i})_\infty}\frac{(q^{1+k(n-i)+\mu_i})_\infty}{(q^{m+k(n-i)+\mu_i})_\infty}\right\} \cdot \prod_{1\le i<j\le n}^{k,\aleph_\lambda}\left\{\frac{(q^{k(j-i)+\mu_i-\mu_j})_\infty}{(q^{1+k(j-i)+\mu_i-\mu_j})_\infty}\frac{(q^{m+k(2n-i-j)+\mu_i+\mu_j})_\infty}{(q^{1+m+k(2n-i-j)+\mu_i+\mu_j})_\infty} \cdot \frac{(q^{1+k(j-i-1)+\mu_i-\mu_j})_\infty}{(q^{m+k(1+2n-i-j)+\mu_i+\mu_j})_\infty}\frac{(q^{1+m+k(-1+2n-i-j)+\mu_i+\mu_j})_\infty}{(q^{k(j-i+1)+\mu_i-\mu_j})_\infty}\right\}$$

With this specialization, Remark 3.20 shows that all the conditions of the multilateralization lemma 4.3 are satisfied. Applying this lemma, therefore, yields the identity

$$(q)_\infty^n \prod_{1\le i<j\le n} \frac{(q^{k(j-i)})_\infty}{(q^{k(j-i+1)})_\infty} = \sum_{\mu\in\mathbb{Z}^n} (-1)^{|\mu|} q^{-kn(\mu)+3n(\mu')+|\mu|(m+k(n-1)+1)} \cdot \prod_{i=1}^{n}{}^{m,1} \left\{ \frac{(q^{1+k(n-i)+\mu_i})_\infty}{(q^{m+k(n-i)+\mu_i})_\infty} \right\} \cdot \prod_{1\le i<j\le n}{}^{k,1} \left\{ \frac{(q^{1+k(j-i-1)+\mu_i-\mu_j})_\infty}{(q^{m+k(1+2n-i-j)+\mu_i+\mu_j})_\infty} \frac{(q^{1+m+k(-1+2n-i-j)+\mu_i+\mu_j})_\infty}{(q^{k(j-i+1)+\mu_i-\mu_j})_\infty} \right\} \tag{4.75}$$

Finally, mimicing the proof of the classical Euler's Pentagonal Number Theorem and setting $m = 0$ (i.e., $t = q^k$ and $b = q^{2k(n-1)}$) gives the desired identity. Notice that the resulting identity (4.71) could also be written in the form

$$\prod_{i=1}^{n} \left\{ (q)_\infty (1-q^{ki})^{n-ki} \right\} = \sum_{\mu\in\mathbb{Z}^n} (-1)^{|\mu|} q^{-kn(\mu)+3n(\mu')+|\mu|(k(n-1)+1)} \cdot \prod_{1\le i<j\le n} \left\{ \prod_{r=0}^{2k-2} (1-q^{r+1+k(-1+j-i)+\mu_i-\mu_j})(1-q^{r+1+k(-1+2n-i-j)+\mu_i+\mu_j}) \right\} \tag{4.76}$$

using the standard product notation instead. □

Remark 4.12. First note that, as in Remark 4.5, the result may be extended into negative integers m and k with some extra work.

Similar to the alternative forms (4.66) and (4.67) of the D_n and B_n Rogers–Ramanujan identities, D_n Euler's Pentagonal Number Theorem (4.71) can be written as a determinant identity, in particular, in the case $k = 1$.

Applying same steps in the proof of Remark 4.10 to the $k = 1$ instance of (4.71) gives the identity

$$2\,(-1)^{\binom{n}{2}} q^{-n(n+1)(2n+1)/6}\, \theta(q;q^3)^n \prod_{1\le i<j\le n} \frac{(q^{j-i})_\infty}{(q^{j-i+1})_\infty} = \det_{1\le i,j\le n} \left(q^{(n-i)(j-1)} \theta(q^{(2n-2i+j)};q^3) + q^{-(n-i)(j-1)} \theta(q^{(2n-2i-j+2)};q^3) \right) \tag{4.77}$$

The identity (4.77) does not, again, give rise to new relations between theta functions. Yet, it appears to be a new determinant identity involving theta functions. It should be also noted that the identity can be put into different forms by transposing matrices, etc.

Euler's Pentagonal Number Theorem and Rogers–Ramanujan identities correspond to the first and second iteration of the Bailey Lemma. In general, iterating Bailey Lemma N times gives rise to so–called generalized Watson transformation and a limiting case of it yields the generalized Rogers–Selberg identity. The Jacobi triple product identity is then used to compute a product representation under certain specializations in the classical case. The resulting identities are called the (extreme cases of) Andrews–Gordon identities which generalize the Rogers–Ramanujan identities. The full Andrews–Gordon identities may be proved using the classical two parameter Bailey Lemma [1].

A multiple analogue (3.64) of the generalized Watson transformation is already obtained above by iterating the BC_n Bailey Lemma 3.17. A limiting case of this transformation gives a multiple extension of the extreme cases of Andrews–Gordon identities similar to the Rogers–Ramanujan identities proved in this paper. Both the extreme cases and a remarkable full version of Andrews–Gordon identities are investigated in another publication [16].

The BC_n Bailey Lemmas 3.17 and 3.8 may be used to prove generalizations of many important identities as in the classical case. Elliptic and basic (trigonometric) root system analogues of several significant classical summation and transformation identities are proved using these results in [15] (also see [35], [36], [42], [38]). The classical one parameter Bailey Lemma has another extension known as WP–Bailey Lemma [3], [6], [44]. An interpretation of the elliptic BC_n Bailey Lemma 3.8 in the setting of a multivariate interpolation problem is used in [17] to obtain what is called interpolation BC_n Bailey Lemma, which may be considered as a multiple elliptic generalization of the WP–Bailey Lemma.

References

[1] A. K. Agarwal, G. E. Andrews and D. M. Bressoud, *The Bailey lattice,* J. Indian Math. Soc. **51** (1987), 57–73.
[2] G. E. Andrews, *On the proofs of the Rogers–Ramanujan identities,* IMA volumes in Math. and its app., (D. Stanton ed.), Vol 18 (1989), 1–13.
[3] G. E. Andrews, *Bailey's transform, lemma, chains and tree,* in Special functions, (J. Bustoz et al. eds.), 2000, pp. 1–22.
[4] G. E. Andrews, *Umbral calculus, Bailey chains and pentagonal number theorems*, J. Comb. Th. (A), **91** 464-475 (2000).
[5] G. E. Andrews, *The Theory of Partitions*, Encycl. Math. and Its Application, Vol. 2, G.- C. Rota, ed., Addison-Wesley, Reading, 1976, (Reissued: Cambridge University Press, London and New York, 1985).
[6] G. E. Andrews and A. Berkovich, *The WP-Bailey tree and its implications*, J. London Math. Soc (2) **66** (2002), 529–549.
[7] G. E. Andrews, A. Schilling and S. O. Warnaar, *An A_2 Bailey lemma and Rogers–Ramanujan–type identities,* J. Amer. Math. Soc. **12** (1999), 677–702.
[8] W. N. Bailey, *Identities of the Rogers–Ramanujan type,* Proc. London Math. Soc. (2) **50** (1949), 1–10.
[9] A. Berkovich and F. G. Garvan, *Some observations on Dysons new symmetries of partitions*, Journal of Combinatorial Theory, Series A, to appear, arXiv:math.CO/0203111.
[10] A. Berkovich and P. Paule, *Variants of the Andrews-Gordon Identities*, submitted to Ramanujan J., arXiv:math.CO/0102073.
[11] D. M. Bressoud, *A matrix inverse,* Proc. Amer. Math. Soc. **88** (1983), 446–448.
[12] D. M. Bressoud, *The Bailey lattice, an introduction,* in Ramanujan revisited, (G. E. Andrews et al. eds.), Academic Press, NewYork, (1988), 57–67.
[13] H. Coskun, *A BC_n Bailey lemma and generalizations of Rogers–Ramanujan identities,* August 2003, Ph.D. thesis.
[14] H. Coskun and R. Gustafson, *Well–Poised Macdonald Functions W_λ and Jackson Coefficients ω_λ On BC_n,* Contemporary Mathematics, AMS, to appear, arXiv:math.CO/0412153.
[15] H. Coskun, *Elliptic and basic hypergeometric series summation and transformation identities associated to root systems*, in preperation.
[16] H. Coskun, *Andrews–Gordon Identities associated to root systems*, in preparation.
[17] H. Coskun, *Interpolation Bailey Lemma,* in preparation.
[18] F. J. Dyson, *A new symmetry of partitions,* J. Combin. Theory, **7** 56-61, 1969.
[19] Euler, L. *Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)$ etc. in seriem simplicem.,* Acta Academiae Scientarum Imperialis Petropolitinae 1780, pp. 47-55, 1783.

[20] L. B. Frenkel and V. G. Turaev, *Elliptic solutions of the Yang–Baxter equation and modular hypergeometric functions,* The Arnold–Gelfand Mathematical Seminars, Birkhauser, Boston, MA (1997), 171–204.
[21] K. Garret, M. E. H. Ismail and D. Stanton, *Variants of the Rogers–Ramanujan identities,* Adv. in App. Math. **23** (1999), 274–299.
[22] G. Gasper and M. Rahman, *Basic hypergeometric series, Encyclopedia of mathematics and its applications,* Vol 35 Cambridge University Press, Cambridge, (1990).
[23] R. A. Gustafson, *The Macdonald identities for affine root systems of classical type and hypergeometric series very–well–poised on semisimple Lie algebras,* Ramanujan International Symposium on Analysis, (December 26–28, 1987, Pune, India) (N. K. Thakare, ed.), (1989), 187–224.
[24] G. H. Hardy, *Ramanujan,* Cambridge Univ. Press, Cambridge, 1940.
[25] J. E. Humphreys, *Introduction to Lie algebras and representation theory,* Springer–Verlag, Berlin, (1972).
[26] C. Krattenhaler, *Advanced determinant calculus,* Sminaire Lotharingien Combin. 42 ("The Andrews Festschrift") (1999).
[27] G. M. Lilly and S. C. Milne, *The A_ℓ and C_ℓ Bailey transform and lemma,* Bull. Amer. Math. Soc. (N. S.) **26** (1992), 258–263.
[28] I. G. Macdonald, *Affine root systems and Dedekind's η function,* Invent. Math. **15** (1972), 91–143.
[29] I. G. Macdonald, *The poincare series of a coxeter group,* Math. Ann. **199** (1972), 161–174.
[30] S. C. Milne, *The C_ℓ Rogers–Selberg identity,* SIAM J. Math. Anal. (2) **25** (1994), 571–595.
[31] W. G. Morris, *Constant term identities for finite and affine root systems: Conjectures and theorems,* Ph.D. dissertation, Univ. of Wisconsin–Madison, (1982).
[32] A. Okounkov, *On Newton interpolation of symmetric functions: A characterization of interpolation Macdonald Polynomials,* Adv. in Appl. Math. **20** (1998), 395–428.
[33] E. Rains, *BC_n–symmetric abelian functions,* math.CO/0402113.
[34] L. J. Rogers, *Second memoir on the expansion of certain infinite products,* Proc. London Math. Soc. **25** (1894), 318–343.
[35] H. Rosengren, *Elliptic hypergeometric series on root systems,* Adv. Math. **181** (2004), 417-447.
[36] H. Rosengren and M. Schlosser, *Summations and transformations for multiple basic and elliptic hypergeometric series by determinant evaluations,* Indag. Math. **14** (2003), 483-514.
[37] A. Schilling and S. O. Warnaar, *A higher level Bailey lemma: proof and application*, The Ramanujan Journal **2** (1998), 327–349.
[38] M. Schlosser, *Summation theorems for multidimensional basic hypergeometric series by determinant evaluations*, Discrete Math. **210** (2000), 151-169.
[39] L. J. Slater, *Further identities of the Rogers–Ramanujan type,* Proc. London Math. Soc. (2) **54** (1952), 147–167.
[40] J. Stembridge, *Hall–Littlewood functions, plane partitions, and the Rogers–Ramanujan identities,* Trans. Amer. Math. Soc. **319** (1990), 469–498.
[41] S. O. Warnaar, *50 Years of Bailey's lemma,* Algebraic Combinatorics and Applications, 333-347, (Springer, Berlin, 2001).
[42] S. O. Warnaar, *Summation and transformation formulas for elliptic hypergeometric series,* Constr. Approx. **18** (2002), 479–502.
[43] G. N. Watson, *A new proof of the Rogers–Ramanujan identities,* J. London Math. Soc. **4** (1929), 4–9.
[44] V. P. Spiridonov, *An elliptic incarnation of the Bailey chain*, Internat. Math. Res. Notices, no. **37** (2002), 1945–1977.

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