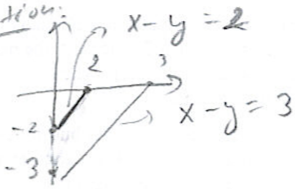
Ex Evaluate the integral

⊙ $\iint_R e^{\frac{(x-y)(x+y)}{(x-y)^2}}\, dA$ R is a trapezoidal region with vertices $(2,0)$, $(3,0)$, $(0,-2)$, $(0,-3)$

Solution:



make a change of the variables using $u=x+y$ $v=x-y$: T^{-1}

we select this change using ⊙ $x=\frac{1}{2}(u+v)$ $y=\frac{1}{2}(u-v)$: T

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = +\frac{1}{2}$$

find the region S in uv-plane

$x=2$ $y=0$ $2=\frac{1}{2}(u+v)$ $0=\frac{1}{2}(u-v) \Rightarrow u=v \Rightarrow u=2=v$

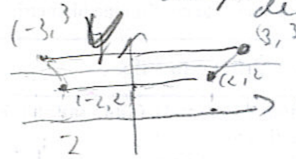
$x=0$ $y=-2$ $0=\frac{1}{2}(u+v)$ $-2=\frac{1}{2}(u-v) \Rightarrow u=-v \Rightarrow v=2,\ u=-2$

$\Rightarrow$ the line $(2,0)(0,-2) \longrightarrow (2,2)\ (-2,2)$

analogously $(3,0)(0,-3) \longrightarrow (3,3)\ (-3,3)$

the points $(2,2),(3,3)$ define the line $\Rightarrow u=v$

the points $(-2,2),(-3,3)$ define the line $\Rightarrow u=-v$



$$\iint_R e^{\frac{(x-y)(x+y)}{(x-y)^2}}\, dA = \iint_S v^2 e^{v\cdot u} \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv =$$

$S=\{(u,v)\mid 2 \le v \le 3\ ;\ -v \le u \le v\}$

$$= \int_2^3 \int_{-v}^{v} v^2 e^{vu} \left(-\frac{1}{2}\right) du\, dv = -\frac{1}{2}\int_2^3 \frac{v^2}{v} e^{v\cdot u}\Big|_{-v}^{v} dv =$$

$$= -\frac{1}{2}\int_2^3 v e^{v^2} dv + \frac{1}{2}\int v e^{-v^2} dv = -\frac{1}{4}\int_4^9 e^w dw - \frac{1}{4}\int_{-4}^{-9} e^z dz$$

$v^2 = w \Rightarrow 2v\, dv = dw$

$-v^2 = z \Rightarrow -2v\, dv = dz$

You have to complete the calculations in solving the two integrals.